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THE MATHEMATICS TEACHER

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THE MATHEMATICS TEACHER

Volume XXXI Number 5

Edited by William David Reeve

Deferred Arithmetic*

By G. T. Buswell
University of Chicago

In looking over the program of this convention I was almost startled to read as the title of one paper "What's Right with Education." In these days of adverse criticism of practically everything the schools attempt to do, my admiration goes to the man who is willing to defend some of their positive achievements. I almost wish that my topic this morning could be "What's Right with Arithmetic" rather than the topic assigned to me, which, in its very nature, focuses attention on those defects in arithmetic which might justify deferring the subject.

Preliminary to any useful discussion of the topic it is wise to clarify the issue. Although the proposal to defer parts of arithmetic has been made periodically for a number of decades, the present rather widespread interest was no doubt stimulated very directly by the series of three articles written by Superintendent Benezet and published in the *Journal of the National Education Association* in 1935 and 1936. In the first of these articles Mr. Benezet expresses his belief as follows: "If I had my way, I would omit arithmetic from the first six grades. . . . The whole subject of arithmetic could be postponed until the seventh year of school and it could be mastered in two years' study by any normal child." This downright statement which appears to be positive and clear, is then qualified in a surprising fashion, surprising at least to one who has never lived in New England. Mr. Benezet proceeds to say that, while he would not teach arithmetic, he would see that children had many experiences with number. In fact, in his second article he states rather specifically what he would have them do instead of study arithmetic. For example, in Grade I he would have children taught to recognize and read numbers up to 100; he would give them basic ideas of comparison and estimate through the understanding of such contrasting terms as more and less, many and few, higher and lower, etc.; he would teach them to use the calendar. In Grade II he would introduce telling time by teaching children to recognize hours and half-hours; he would use the number system in relation to the paging of a book and in counting scores in games; he would have them learn the meaning of such terms as *half, double, twice, three times;* he would also teach the meaning of *penny, nickel, dime,* and *dollar.* In fact, as Mr. Benezet proceeds through Grades III, IV, V, and VI he outlines what I should consider

* Paper read at the Annual Meeting of the National Council of Teachers of Mathematics at the Hotel Traymore in Atlantic City, N. J. on Saturday, February 26, 1938.

some of the most valuable parts of the arithmetic program. By the time I had finished reading all three of Mr. Benezet's articles I found myself agreeing with him quite completely, with two qualifications. First, the quantitative experiences which he retains after deferring arithmetic until the seventh grade are the very things which are essential to any understanding of arithmetic. Second, if formal arithmetic means to him what it means to me, then, instead of deferring it until the seventh grade, I should like to eliminate it altogether and to substitute in its place an arithmetic program devoid of formality and impregnated with meanings which are understood clearly by the children. Arithmetic should be systematic, at least as viewed by the curriculum-maker and the teacher, but it should not be formal. Systematic arrangement and formality are not synonymous terms. The term "formal" relates to an attitude of mind which may occur in incidental as well as in systematic instruction.

From the preceding statement it seems to me that the topic of this paper must include two items, first, what kind of arithmetic should be taught, second, how should this content be organized and in what grade should the various topics be placed. When attacked from this point of view it may appear that certain parts of arithmetic should be deferred, that other parts should appear earlier, and that still other parts should be omitted altogether. In view of the fact that in Grades IV, V, and VI the number of failures in arithmetic is exceedingly high, it seems particularly clear that some of the content of this area must either be deferred to later grades, or be made easier by building a better foundation at the lower levels of school. It is obviously quite inadvisable to add to the difficulties of the middle grades and consequently the main problem of this paper will need to be referred to the lower grades and to the upper grades. I propose to treat these in this order.

The Arithmetic of the First Three Grades

Without any doubt there has been in the last few years a widespread rebellion against formal arithmetic in the lower grades. I share completely the feeling that the customary method of introducing arithmetic has been far from satisfactory. On the other hand, I am quite aware, as are you, that careful investigations of number abilities and interests of young children reveal a surprisingly large content, so large in fact, that, were the school to defer completely its treatment of arithmetic, the children would, of their own accord, necessarily build up some type of experience in this field. Furthermore, in a world such as ours it is impossible to defer children's quantitative experiences or to defer the great quantity and variety of contacts which they have with numbers and their applications. No one who has read carefully the reports of children's number abilities, such as have been made by Woody, by Buckingham and McLatchy, by McLaughlin and others, can defend the notion that here is an area of experience that the school can simply ignore or defer teaching relating to it until some convenient later time.

The difficulties which have been encountered in the lower grades have been largely due to the application of a superficial psychology of arithmetic. Number concepts are abstract. They represent one of the highest intellectual achievements of the human race. The schools have taken the end-products of these abstractions and have presented them in formal fashion to children in the lower grades. When I use the term "formal" I mean that the content of arithmetic has been presented without due regard to the sense of meaning which it possesses for the child.

The human race built up its mastery of number only after long ages of endeavor. The stages through which the human mind has passed from the time that number experiences were carried on as a one-

to-one relationship to the development of abstract number processes such as we now use give some indication of the difficulty of going from concrete experience to abstract generalization. Instead of giving children the opportunity of building number concepts through many concrete experiences, the school has attempted to short-circuit this process by simply having children memorize the number combinations as their introduction to arithmetic. The situation has been aggravated by the multiplicity of formal arithmetic tests and drill exercises which have given children a set of verbal symbols almost devoid of meaning. It is not surprising that children who have been forced to learn their number combinations through drill and who have then been drilled on the four fundamental processes with whole numbers and fractions should have trouble in understanding what it is all about, or should do many of the absurd things which Superintendent Benezet found them doing.

The number relations with which arithmetic deals are highly abstract. Furthermore, the values of the arithmetic processes correspond to the degree of their abstraction. The very essence of arithmetic resides in the use of these abstract processes in relation to social situations. However, neither vagueness nor formality is a necessary characteristic of abstractions. When taught in a formal manner, the abstractions of arithmetic may result in a set of verbal generalizations which have little meaning to the child and which cannot be applied with any degree of genuineness. On the other hand, if the abstractions in arithmetic are made to grow out of the concrete experiences of children and if the relationships which they embody are developed so gradually that at every stage the child fully understands what is done, arithmetic may still be abstract but be perfectly clear and meaningful when applied to social situations.

It is in this very process of building clear foundations for abstract understandings that the arithmetic for the primary grades has its greatest opportunity. Children already have all the quantitative experiences that are necessary to develop a number concept from the primitive one-to-one stage to the stage of meaningful abstraction. For example, a small child will divide a sack of cookies among friends by saying, "One for you, and one for you, and one for me. Another for you, another for you, and another for me," until the cookies are all gone. Soon, however, he will learn that there is a superior way to deal with cookies and he will find that by counting the number of children and counting the number of cookies he may short-cut the primitive one-to-one method of dividing. Or again, a child may learn the difference between ordinals and cardinals by having it explained to him in formal terms but he will never understand it if he learns it that way. On the other hand, he may be led to learn from his own experiences that a child cannot be third in line unless there are at least three children, that there cannot be a fifth day in the month until there are at least five days.

The various number facts may be taught in such formal fashion that they will be learned simply as verbal statements, true because the book or the teacher says they are true. On the contrary, the relationships among numbers may be developed through experiences so concrete in character that when the child finally arrives at such a generalization as 4 plus 5 is 9, it will be not simply a verbal statement but an item of his own experience which he is willing to vouch for and defend against all odds, simply because he knows that four things and five more things always make a total of nine things. That type of understanding does not come from drill on number combinations. It comes rather from much concrete experience in combining numbers.

The essential point which I want to stress here is that the arguments for deferring arithmetic are all based on the sad results of a practice of teaching number

combinations and number relations as abstract verbal statements devoid of concrete background. The way Thorndike's Law of Exercise has been abused in arithmetic practice exercises is a conspicuous example of the extent to which formalism can be carried. I know of no educational psychologist who would be willing to defend the extent to which memory drills have been carried in the primary grades. This type of formal arithmetic should not merely be deferred; it should be completely abandoned. However, eliminating formal arithmetic from the primary grades does not mean that there shall be no arithmetic taught at that level. Instead, it provides an opportunity for developing through many concrete experiences genuine foundations for quantitative thinking.

There is available now a considerable body of data showing how the school may substitute meaningful social arithmetic in place of the formality of number drills. If the school would emphasize an objective of understanding arithmetic as much as it formerly emphasized speed in computation, there would be an opportunity for laying a foundation in a type of arithmetic which could profoundly affect the subsequent quantitative thinking of children. Arithmetic should not be deferred beyond the primary grades; it should be properly selected and organized to build on the interests and needs which research has shown children possess at the age of entering school. This type of arithmetic should give meaning to normal social experiences. It is the schools' best defense against abstract verbalism and meaningless formality in the middle and upper grades.

The Upper Period

The program of upper-grade arithmetic exhibits considerable confusion at the present time. This is due, in part, to the fact that in those schools which maintain the traditional eight-year organization the customary seventh- and eighth-grade topics have ordinarily been retained, whereas in those schools which are organized under the junior-high-school plan the arithmetic of the seventh and eighth grades has been influenced to a much greater extent by the algebra and geometry of the high-school period. This variation in content has given rise to two distinct types of textbooks. Furthermore, it has emphasized the notion that arithmetic should be completed by the end of the sixth grade and that the child is then ready for "higher" forms of mathematics.

The proposal which I should like to suggest and defend is that arithmetic has not been, and probably cannot be, satisfactorily completed by the end of the sixth grade. Furthermore, there are certain topics in arithmetic which may be extended or deferred not only up to the seventh and eighth grades but may even be integrated with profit into high-school mathematics.

The theory of social utility has, in general, affected arithmetic in a negative fashion, causing a reduction in the content of the subject. In most, although not all cases, it has been applied only to the computational aspects of arithmetic. It has been assumed that unless computational processes are found to be commonly used there is no particular need for the school to teach them.

Since the report of the National Society Yearbook Committee in 1930 there has been a considerable change in attitude toward the reduction of the arithmetic program. This change was due not so much to the report of the original Yearbook Committee as to the report of Dr. Brueckner's reviewing committee which appeared as the last chapter of the 1930 Yearbook. This reviewing committee did an excellent service to education in focusing attention on the noncomputational objectives of arithmetic. Certainly, since 1930 there has been a marked increase in interest in social arithmetic and in the use of arithmetic in quantitative but noncomputational situations. Many of these noncomputational uses are highly im-

portant in high-school and adult experiences. In fact, the opportunities for integrating arithmetical thinking into the high-school program are far greater than have been realized up to the present time.

If society is to make the most of the social applications of arithmetic it will be necessary to have a better understanding of arithmetic than the present program affords. One or two illustrations may indicate what I mean. It is desirable that automobile license numbers be recognized quickly and memorized easily. In those states where the total number of cars is small a number system can be devised which seldom goes beyond five or six digits. However, in the states where the number of cars is large the number of digits on the automobile license plate often extends into six or seven places. These long numbers are hard to recognize quickly and are difficult to remember. Consequently, certain states have improved upon the ordinary method of counting and have devised a system of supplementing digits by letters so that they can care for license numbers extending up into the millions by using only five letters and digits combined on a license plate. This combination is simply an extension of the idea of a number system. Few children, or even adults, for that matter, understand how to do this; yet the social value of such an efficient system of numbering is very great. Automobile licenses furnish only one of many examples of the advantages of improving upon the usual system of consecutive enumeration. Under the ordinary program of arithmetic such a topic can hardly be introduced because the course of study is so crowded but, if certain topics could be deferred until children have more mature experience, a positive and worthwhile enrichment of the arithmetic program could easily follow. The system of decimal indexing in public libraries is another example of a highly useful social application, the arithmetic of which is seldom understood.

The essence of arithmetic resides in the systematic character of the number system and in the many types of abstract relationships which exist among numbers. There are many useful concepts related to our number system which could be added in a stretched-out or deferred course of study but which it is impossible to include in a program of arithmetic cramped into six, or even eight, years. The situation in arithmetic has been rendered increasingly difficult by the tendency to push down into the seventh and eighth grades more and more material which was formerly reserved for the courses in algebra and geometry. This paper is not to be interpreted as being critical of an attempt to correlate mathematics but instead to be critical of the attempt to correlate it at the expense of arithmetic. If topics from algebra and geometry are to be pushed down into the grades on the grounds that they are simple and can be understood by children, it is equally logical to propose that difficult arithmetical concepts be deferred until later in the school program, even possibly to the high-school program, for the sake of added understanding and possible integration at that level.

The program proposed here in one which provides a rich mathematical experience and which affords an adequate basis for the understanding of the quantitative aspects of living. It is concerned as much with noncomputational arithmetic as with computation. It sees in the enrichment of quantitative thinking an area of service seldom covered by arithmetic.

It is difficult to achieve a better program of arithmetic. Courses of study set the pattern for textbooks which tend to crystallize things as they are. We have witnessed several serious efforts to study deferring arithmetic, ordinarily referred to as studies of grade placement. The work of the Committee of Seven is perhaps most widely known. It was done carefully by competent people and I have no disposition to be too critical of their efforts. Still, I attach little value to their

findings because the entire scope of their endeavor was limited to the program of arithmetic as it was. It seems to me that arithmetic needs something more than a re-allocation of the grade placement of the topics of arithmetic as now presented.

Summary

The position taken in this paper may be summarized as follows.

First, simply deferring arithmetic as now taught until later grade levels is a purely negative and hopeless undertaking. Arithmetic needs overhauling as well as deferring.

Second, the proposal to defer arithmetic has been widely interpreted to mean no arithmetic at all in the primary grades and, in some places, no arithmetic until a still later school grade. This proposal goes directly against all the research relating to the number experiences and interests of young children. The proponents of deferred arithmetic are making no positive contributions to a constructive program of arithmetic. As opposed to this view I am proposing a program of arithmetic in the primary grades developed completely from the concrete experiences of children and focused primarily upon understanding basic quantitative relationships which are within the experience of young children. Such a program may, if thoughtfully worked out, provide a rich meaningful number experience in the place of the formal number drills now devoted to number combinations and processes.

Third, the development of such a concrete program at the primary level would provide an understanding which would make the work of the middle grades less difficult and would make unnecessary the deferring of some topics which are now being pushed farther ahead.

Fourth, at the upper grade level certain topics in arithmetic may profitably be deferred and presented during the high-school period. There is no reason *per se* why arithmetic is an elementary school subject and entirely out of place in the high school. It is only convention which restricts it to such a position. Much of the work in measurements could more profitably be presented during the high-school period where it can be integrated with the science program. Many problems, such as investments, taxation, and insurance, can be treated much more richly in a later period and can be related in a meaningful way with the social studies program. The arithmetic of simple statistics is exceedingly useful but can be handled better later than the eighth grade. Still more, there is a whole series of extensions of the concepts related to the number system and short-cuts with the various number processes that are difficult to make meaningful to children with immature experience but which might, if deferred until the high-school level, be made of great social significance and of as much value as many topics in algebra or geometry.

Deferring arithmetic need not be considered purely in terms of deferring the topics which are now present. Rather, if some of the time devoted to this subject could be deferred, it would make possible an intellectual enrichment of the course and would make possible many useful social applications.

Preliminary Report of the Joint Commission!

The Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics on "The Place of Mathematics in Secondary Education" is issuing a Preliminary Report in two parts. The first part has already been issued, and it is expected that the second part will appear about July 1st. The report is being issued in a preliminary form in order to secure the comments and suggestions of teachers, administrators, and educators. Persons desiring to receive copies can do so by sending ten cents (coins preferred) for each part to the Chairman, Professor K. P. Williams, Indiana University, Bloomington, Indiana.

Preparation For Teaching Secondary-School Mathematics

By HENRY H. HAGEN, *Principal*
Crane Technical High School, Chicago
and
NORMAN L. SAMUELSON, *Director of Guidance*
Crane Technical High School, Chicago

OVER FIVE HUNDRED principals and teachers of mathematics in one hundred secondary schools in twenty states agree that the successful teacher of mathematics is usually one whose training and experiences have been extensive, one who is not merely a specialist in the field of mathematics, as the child—not the subject-matter—is the first consideration here, as in all fields. The basis for this statement results from a recent investigation made by the *Commission on Unit Courses and Curricula* of the North Central Association of Colleges and Secondary Schools, a preliminary report of which has been previously published.* The generalization was drawn from responses to questionnaires submitted by this group of educators.

The training-pattern for prospective teachers of secondary-school mathematics should consist of six distinct parts, namely:

(1) a broad general education;
(2) Intense training in the field of mathematics;
(3) training in closely related minors;
(4) general and special courses in professional education;
(5) internship, or cadeting under a successful teacher of mathematics; and,
(6) many experiences of a non-academic nature as the sources from which applications can be drawn, thereby presenting the possibilities of an enriched and more interesting presentation.

Each of these six elements should make its contribution toward the development of certain understandings and appreciations, qualities, habits, and attitudes that form the basis for a pleasing personality.

BROAD GENERAL EDUCATION

The prospective teacher of secondary-school mathematics should obtain a broad general education in the major fields of knowledge in order that he may bring into the classroom a breadth of interest compatible with present day society. Ninety-four per cent of the respondents favored such a background, including appropriate courses selected from the fields of the biological sciences, physical sciences, social sciences, art, music, literature, religion, philosophy, and ethics.

If the great purpose of education is to teach students how to live happy, useful, and harmonious lives, then surely, the best teacher is the one who is best able to inspire an appreciation of the fine things in life while presenting the materials of her own particular field. No teacher can do justice to any one subject unless he can fit it into the whole eductional plan. To do this he must have knowledge, understanding, and appreciation of other subjects in the curriculum. This general knowledge will also make his a more interesting personality.

The responding principals and teachers of mathematics are fully cognizant of the influence that breadth of training, or lack

* H. H. Hagen, "Judgments of High School Teachers Concerning the Academic Preparation of Secondary School Teachers," *North Central Association Quarterly*, Vol. XII, No. 2, October, 1937. The complete report for all fields will be published in the above *Quarterly* for April, 1938.

of it, can have upon their teaching ability. They realize, for instance, that:

(1) a general education is the necessary foundation for all specialization—that with such a background they can better orient mathematics in the broad field of education;
(2) that they will assume a less narrow outlook as to the relative importance of mathematics as compared with other subjects in the curriculum;
(3) it opens fields from which enriching materials may be brought into their teaching; and,
(4) they can better correlate mathematics with other subjects in the curriculum.

There is a decided danger, so many respondents stated in subjective comment, that the teacher who possesses a highly specialized knowledge of mathematics, with no broad background, will fail to observe the role that mathematics plays in the whole educational scheme. Without a broad general knowledge and a consciousness of those factors which contribute to social well-being, the teacher may neglect the broader aspects of the educational process.

Training in Mathematics

The prospective teacher of secondary-school mathematics must have a wide understanding of the various levels in this field. He must understand the relationship between the mathematics that he is attempting to teach and those in lower grades in order to sustain a continuity of training, and not risk the formation of gaps which will leave the student struggling along to prevent being a failure. There is a danger that the poorly trained teacher will have such an effect upon the student, making the subject distasteful and difficult. In order to gain the point of view necessary for successful teaching and make his teaching purposeful and effective he must have had definite training in several college courses. In fact, of the principals and teachers responding, 98% advocate that training be obtained in college algebra; 97% in plane trigonometry; 91% in solid geometry; 91% in analytical geometry; 89% in history of mathematics; 82% in differential calculus; and 80% in integral calculus. Other subjects are advocated to a lesser extent. Sixty-five per cent of the respondents would include theory of equations; 49% descriptive geometry; and 46% spherical trigonometry. This training will bring into his teachings the necessary understandings and appreciations of the relationship between his work and that of the next higher level, affording him an opportunity to not only obtain an insight of elements that are needed later, but also obtain a point of view concerning his own teaching subjects which will be invaluable.

Training in Related Subjects

A teacher trained and interested in mathematics has an understanding and appreciation for the subject which a teacher whose major field of interest is elsewhere does not have. His knowledge and background gives him many sources from which to draw in order that he may inspire and motivate his teaching. Consequently, it is desirable to assign all classes in mathematics to one or more teachers in this field whenever such arrangements will provide full teaching loads. The teacher is best fitted to work in the field in which he is best prepared—both in formal training and non-academic experiences. Too many men and women are teaching mathematics without a definite understanding of the value and use of the subject. These teachers are not interested in mathematics, cannot orient it within the educational plan, and should not be teaching it.

On the other hand, the preparation for teaching should not be too highly specialized. There is the danger that a one-sided training will lead to a one-sided or narrow point of view regarding the impor-

tance of mathematics in the curriculum. It will tend to make the type of instruction too theoretical, omitting the more definite ways in which mathematics is applied in the modern world.

Then too, full teaching loads in mathematics are not always available, making it necessary for the mathematics specialist to instruct classes in other subjects, Recommendations, as to desirable teaching combinations with mathematics as the major, were made by the teachers in these one hundred schools. When full teaching programs in mathematics are not available, they would complete the load as follows: 47% would make it general science; 45, physics; 13, English; 13, chemistry; 5, history; and 3, mechanical drawing. Closely related subjects are most desirable—preferably courses in the physical sciences. Excellent applications of mathematics are found in this field, thus making a high degree of correlation between the two advisable. In fact, of the teachers solicited 95% advocate that training be obtained in physics; 80% in chemistry; 70% in general science; 39% in geography; 38% in geology; and 25% in astronomy.

Many other courses in the sciences and applied mathematics are advised. Surveying, navigation, statistics and graphics acquaint the teacher with many interesting applications of mathematics. Other courses in these fields recommended more infrequently are: mathematics of finance, slide rule, map reading, business training, accounting, etc.

Every department of mathematics contacted advocated also some training in social studies, including work in such subjects as economics, sociology and statistics, recognizing the possibilities of coordinating parts of these subjects with mathematics.

Professional Education

The respondents indicate that all teachers of mathematics should obtain training in such professional courses as General Methods of Teaching in High Schools, History of Education, Tests and Measurements, General Psychology, and Educational Psychology. They heartily recommend a course in the History of Mathematics to create a background for teaching and an understanding and appreciation of how present day mathematics evolved. Such a course is the source of much interesting and enriching material.

An adequate broad general education, along with sufficient work in the field of mathematics and professional education should result in the acquisition of certain definite skills in teaching; namely, aptness and fertility of illustration, clearness of illustration, keen sensitiveness to misunderstanding on the part of the student, alertness in questioning, alertness in giving problems, ability to develop interests in pupils that will be lasting and carry over to other subjects and to other phases of life, clear perception of ends, the use of facts and laws of mental growth, individual differences, the use of accepted standards in tests and scales, hygiene of school work and equipment, and outstanding techniques in classroom management.

Special courses in the teaching and administration of mathematics are advised. These include such topics as The Teaching of Arithmetic, The Teaching of Algebra, Curriculum Problems in Mathematics, and Demonstrations and Use of Instruments.

Special emphasis should be placed on the application of psychological principles involved in teaching, for, as the teachers comment: many lack an understanding of human nature; many fail to recognize pupil difficulties; many lack an understanding of remedial procedures; and many lack an understanding of pupil limitations. A study of the various investigations which have been reported on difficulties encountered in elementary school arithmetic is also highly recommended, including the ways in which children go

wrong in learning number concepts and operations.

Internship or Practice Teaching

The theory of the classroom should be followed by practice. The prospective teacher should observe superior teachers while at work—to observe the techniques used in the teaching procedure. He should plan units—present them—test—diagnose difficulties and reteach, all under the guidance of an expert.

Non-Academic Experiences

The ideal teacher is a person of broad understandings and appreciations derived from a wide range of experiences. These understandings and appreciations should augment a well-rounded personality which appeals and wins confidence; creates an honest, democratic interest and sympathy for all sorts of social groups; and develops a good sense of values and suspended judgment. Many of these understandings and appreciations relate to subjects which constitute the materials of instruction in the field of mathematics. Many of them can be developed by participating widely in non-academic experiences where mathematics is used in practice, for as many respondents state: they lack experience in industry; they lack familiarity with a range of mathematical applications; engineering knowledge affords fine applications, they found their extent of statistical applications limited; and, feel that surveying furnishes fine applications for geometry.

Summary

By way of summary, the prospective teacher of mathematics should obtain a broad general education upon which to build a comprehensive sequence in mathematics and a closely related minor. He should have much training in professional education, including special methods courses and practice teaching. He should bring many practical applications of mathematics into the classroom from non-academic sources thereby presenting the possibilities of an enriched and more interesting presentation.

Teachers of Mathematics!

Teachers of mathematics will have two important meetings this summer. The first is the meeting of the National Council of Teachers of Mathematics to be held in New York City from June 27th to 29th inclusive, and the second is the semi-centennial celebration of the American Mathematical Society to be held also in New York City from September 6th to 9th inclusive. The program for the National Council meeting will be found on pp. 241–243 of this issue.

At each of the above meetings it will be possible for teachers to see

1. An exhibit of mathematical materials of all kinds at Teachers College, Columbia University.
2. The George A. Plimpton and the David Eugene Smith Collections of rare editions of mathematical books in the old Low Library building at Columbia University.
3. The library of the American Mathematical Society also in the old Low Library at Columbia University.
4. Models and apparatus in the Astronomy Department at Columbia University.

Fugitive Materials

By Frances A. Mullen
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In every issue of our professional journals and from the speakers' table at every professional gathering, the teachers of mathematics of this country are adjured to enrich their courses, to vitalize their classes, to motivate their students, by emphasizing the relations of mathematics to current science, industry, and life, by stressing the history of mathematics and its part in the development of civilization, by introducing intriguing puzzles and subtle humor to revive the flagging spirits of our pupils when boredom threatens.

Enthused, we rush back to our classrooms, determined to carry out the precepts so ably urged upon us. We retell a choice anecdote. If it is successful in arousing a little interest and discussion, we suggest that someone investigate the subject a little further and make an extra-credit report on it as soon as possible; then breathlessly turn to the day's work to make up for the "lost time."

And what happens then? If the subject has been biographical, and if we are lucky enough to have one or two of the most simply written histories of mathematics, together with a good encyclopedia for secondary students, in our school library, the student who has caught our enthusiasm may be able to find help there, without more assistance from his already overburdened teacher.

But suppose the topic leads into one of the fascinating recreations in mathematics, concerning which you remember hazily that there is a good discussion in one of the puzzle books, or in the back of another text than that which you are using, or in an old issue of *The Mathematics Teacher*. Can you either take time to find it for the student yourself, or expect him to find it in the library with only a vague clue? Perhaps your ninth-grade class is approaching a unit on numerical trigonometry and is enthusiastic about your suggestion that they make model transits. Can you locate that article from *Popular Mechanics* which you cut out several years ago, but have never used until last night's lecturer inspired you to attempt once more to "vitalize" your teaching?

Or, most important of all the supplementary topics, and certainly the most difficult to find reference material for, let us assume that you have been discoursing at length on the value of mathematics and its relationships to many phases of modern life and civilization. The rounded periods of last night's orator seem to fall a little flat this morning as you endeavor to translate them into terms understandable to your class. Irrepressible Johnny insists that he certainly can't see any connection between mathematics and music, —and as for art, his Dad's a commercial artist and he never uses any mathematics. Well, at least you have aroused interest and discussion, but the bell is going to ring in sixty seconds, and unless you can refer Johnny and his friends to some material which explains the point simply and with concrete illustration, most of your day's work will have been wasted.

Too often the writers on the enrichment of the mathematics course seem to expect the impossible from the struggling practitioner in the classroom. We have no time nor license to give rounded courses in the history of mathematics or to lecture by the hour on the applications and importance of our subject. We have "ground to cover" or "objectives" to reach,—often enough with plenty of handicaps in the way of over-large classes, groups which are too diversified in ability, inadequate space or equipment, too crowded a school day. We need help, now, not so much in further adjuration to do better (though we admit plenty of room for improve-

ment), as in the preparation by experts of more intriguing materials written directly for high-school students (and not entirely for the most advanced and most intelligent of those), interestingly illustrated, and attractively published.

Fortunately such material is being prepared in increasing quantity. The monograph on *Numbers and Numerals*[1] is one such, and the promise of the National Council of Teachers of Mathematics of more to come is most welcome.

Much of the most useful material is, however, now available only in scattered periodical literature and in advertising brochures and miscellaneous pamphlets, and it is difficult to make such fugitive materials accessible to our students. Becoming increasingly conscious of such difficulties, the writer tried for several semesters to solve the problem by having her classes cooperate in collecting, mounting, and indexing scrap books of such materials. These scrap books proved of value for the students who helped make them, but of relatively little use for reference, due to the difficulty of making and maintaining adequate indexes.

The obvious solution of such a problem is a vertical file. In fact the solution seems so obvious as to need no comment. However, two years of experience with and classroom use of such a file have shown it to have so many values that it seems to merit some description and discussion at this time.

The file now has the following major headings:

1. Biographies
2. Other historical notes
3. Puzzles
4. Other recreations
5. Pictures
6. Plays
7. Club Programs
8. Humor
9. Importance—general
10. Relation to other fields
11. Miscellaneous
12. To be filed.

Sources of material have already in part been indicated. The scrap books of previous semesters have been dismantled. Copies of *The Mathematics Teacher* and other journals have been literally torn apart, and separate articles filed. Puzzles, notes, and comments brought in by the pupils from popular magazines and the newspapers, from *Current Science* and *Believe It or Not*, and from the Sunday supplements, have been included. A considerable number of advertising pamphlets prepared by the Bell Telephone Company, Ford, Westinghouse Electric, Johannsen Gages, Gruen Watches, and others have been found to have value as picturing the growth of industry, illustrating in various ways the importance of mathematics in industry and the increasing reliance on refinements of measurement. Some very good material, interestingly illustrated, has appeared in the advertisements of the publishers of various mathematical textbooks. Even programs and dinner menus from professional gatherings have contained material that found its way into the file.

A student committee of the Mathematics Club was busy last semester going through the textbooks in the classroom cupboard, listing page references to the biographical and historical notes, and to the recreations, fallacies, etc., discussed therein. In Archimedes folder, for example, they placed a sheet indicating that a full page biography and picture would be found on such and such a page of so and so's geometry, that stories about him appeared in various other places. Such topics as magic squares, fallacies, construction projects, directions for making models, were noted and exact references to the page and book were inserted in the file at appropriate points.

There is much material available in the professional literature which is not directly suitable for student use, either because of vocabulary and other difficulties,

[1] David Eugene Smith and Jekuthiel Ginsburg, *Numbers and Numerals*. New York: Bureau of Publications, Teachers College, 1937. Price, 25¢ postpaid.

or because of point of view, but which nevertheless contains some of the most valuable items. An article in itself unsuitable for secondary pupils often gives anecdotes or references to curious phases of mathematics which are very directly appealing to students of this age. Such paragraphs may be marked, and student members of the Mathematics Club committee given the task of copying each on a separate sheet of paper, of devising appropriate titles, and of finding proper places in the file for each.

In the section "Pictures" will be found a wide range of illustrations which might at first sight seem to have little reason for being in a mathematics file. There is a folder for instance on Egypt, containing photographs and drawings taken largely from advertisements, which illustrate costumes and people, the pyramids and the Sphinx, and many other phases of the art, architecture, and industry of that country. Similar folders are provided for Greece, Rome, Babylon, China, Arabia, and Medieval Europe. Actual portraits of famous mathematicians are not included here, being filed with their respective biographical folders. Also there is a group of folders containing striking pictures of modern engineering achievements,—airplanes, trains, bridges, factories, and industrial processes—the qualification for admission of a picture being that it convey a dramatic feeling of the size, complexity, or precision of the subject.

These picture folders are actually used perhaps as often as any others. Every semester students are encouraged to make for extra credit posters illustrating the interrelationship of mathematics with the development of civilization, and other series stressing the importance of mathematics in various phases of modern life. With the file and the pictures available it is now possible to make such an assignment in half of an ordinary class period, whereas in previous years it was necessary to use much more of the precious class time in talking, illustrating, and demonstrating, before the pupils caught the idea and could see their way to fulfilling the requirements. When the assignment is to be made, a few of the best posters from last semester are prominently displayed (if they have not been in evidence most of the semester), a collection of suggestive pictures from the file is arranged on the bulletin board, and a list of possible poster legends is written on the board. These legends include epigrams and quotations on the importance of mathematics or its relation to various phases of life, and, for the more literal-minded child, such phrases as "Mathematics helped build this bridge," "Without mathematics this plane would be impossible" are suggested.

At another time, for the pupil who prefers literary to artistic expression, it is suggested that themes be written on the importance of mathematics in some particular phase of science, industry, or art, and the file again comes into great demand.

If the preservation of these fugitive materials is of value in supplementing the regular class work, the help which they can render in the task of directing a live and vital mathematics club is obvious. A good student committee can find suggestions for programs and develop them with a minimum of help from the teacher when they have access to these materials. In our file, one section is devoted to articles from *The Mathematics Teacher* which suggest possible activities for such clubs, and each semester the newly elected officers and the chairman of the program committee are directed in the first flush of their enthusiasm to go through these articles and select topics and projects which particularly appeal to them.

Another section contains a number of plays. The chairman of the drama committee of the club, who has now for several semesters had the job of selecting one for presentation to the club each term, is this time inspired to attempt to write one herself, drawing on parts of existing ones, but adapting all to her own purposes.

One project of our mathematics club, for which our file of fugitive materials is particularly indispensable, is the publication of a monthly mathematical magazine. This is a four-page mimeographed paper which contains puzzles, cartoons, crossword puzzles involving definitions of mathematical terms, articles on the importance and applications of mathematics, and on historical topics, as well as news of the doings of the Math Club. Obviously the success of such a venture is particularly dependent on the proper construction, upkeep, and use of such a file; but the file should justify the time and effort expended upon it in many other mathematics classrooms.

Few of our classrooms are equipped with a good steel filing cabinet, and most administrators perhaps would be astonished at a request for one. The problem, however, may be solved inexpensively, at least for a start. A fairly strong box, which the office supply companies call a transfer case, to hold folders for ordinary letter size papers may be purchased for about $1.50. A set of tagboard guides for the major divisions and two hundred manila folders of the cheapest grade will probably bring the total expense to approximately five dollars. This is not an impossible sum for a mathematics club to raise by an entertainment or candy sale, and such equipment will be found adequate for a classroom file.

Later, if expansion warrants it, the school administration might be convinced of the necessity of a four-drawer file whose usability and durability would greatly exceed those of the single transfer box. The project might be developed on a departmental scale, and the collection housed in the school library if circumstances indicated that such would be advisable.

For best results, a maximum use should be made of student cooperation and help. When the students raise the money, contribute some of the material, and do a large part of the routine work of preparing and filing the folders, they will feel a continuing interest in the project, and furthermore will be gaining acquaintance with the contents.

The test of the value of the file will come in the amount of student use which it receives. It should contain only materials understandable and of interest to at least some of the students who will use it, and much of use even to the humblest, so that he will not return discouraged, as he too often does from an attempt to penetrate the mysteries of the majority of the available books.

The purpose of the project is not to replace the books on the history of mathematics or on the recreations, but rather to supplement them, and by creating and cultivating student interest, to encourage the use of such books. The file does not give a complete or scholarly or rounded treatment of most topics. It merely takes what comes, and preserves suggestions, and hints which may be encouragements to further study and play.

If it is to be kept a file for the students, it should not take the place of the teacher's personal file. Many suggestions, ideas, and discussions which she may want to preserve do not interest children. Such materials should not be kept in this box; it should be definitely reference material for the pupils, and they should recognize and use it as such, taking pride in adding to it themselves whenever possible.

As mathematics teachers become more aware of the results to be obtained by judicious use of supplementary topics in the enrichment of their basic courses in mathematics, there will doubtless grow up a demand which will be answered by an increased production of permanent materials adapted to high-school students, but there will always be fugitive materials of value. Librarians are learning to preserve and catalogue much such material for the social sciences and other fields. Mathematics teachers will find real pleasure as well as utility in such a collection in their own field.

The Mathematics of the Automobile*

A GOOD poster in the ordinary sense of the word should need no explanation. However, the three educational posters prepared by the Chevrolet Motor Division express thoughts rather than things. This circumstance may make it permissible to elaborate them with a commentary.

These three posters are only part of a larger collection assembled by Dr. W. D. Reeve, Professor of Mathematics at Teachers College, Columbia University to portray the importance of mathematics in civilization and with this thought in mind the drawings should be contemplated.

If we think of the feudal age, we visualize, armored men on horseback; if we think of the twentieth century, we visualize a stream of motor vehicles. This modern age, whether we like it or not, is a motorized civilization. Our posters then give an answer to the question:

"How Important Is Mathematics In Automobile Engineering?" And the answer is:

"Very Important!"

It may be stated here that for the engineer, irrespective of his particular field, mathematics is not quite the abstract science of the pure mathematician. He rather thinks of mathematics in terms of mathematical physics. The relations between physics and mathematics always have been very close. For instance, Leibnitz and Newton created differential and integral calculus as a tool with which to solve certain problems of mechanics. Thus it will not be surprising that these engineering posters are grouped less according to the branches of mathematics than to physical laws and to those functions of a "car" that express these laws best. If then the posters do not specify directly, in which branches of mathematics the automobile engineer is particularly interested, by indirection they point out forcefully that—with the diversity of the problems focused in a motor vehicle—practically all departments of mathematics are useful and essential, from algebra to analytical geometry and from the rule of three to integration. It is the task of the engineer, first to comprehend and circumscribe a problem, then to select that mathematical (or experimental) tool that solves the problem best. Again, it should be noted also that mostly in experimental tests there is needed, in addition to the visible apparatus, a considerable mathematical apparatus.

A motor vehicle must possess, beyond anything else, three attributes: *Performance*, *Power*, and *Strength*. Our posters are composed to express these three requirements. Their headlines read successively:

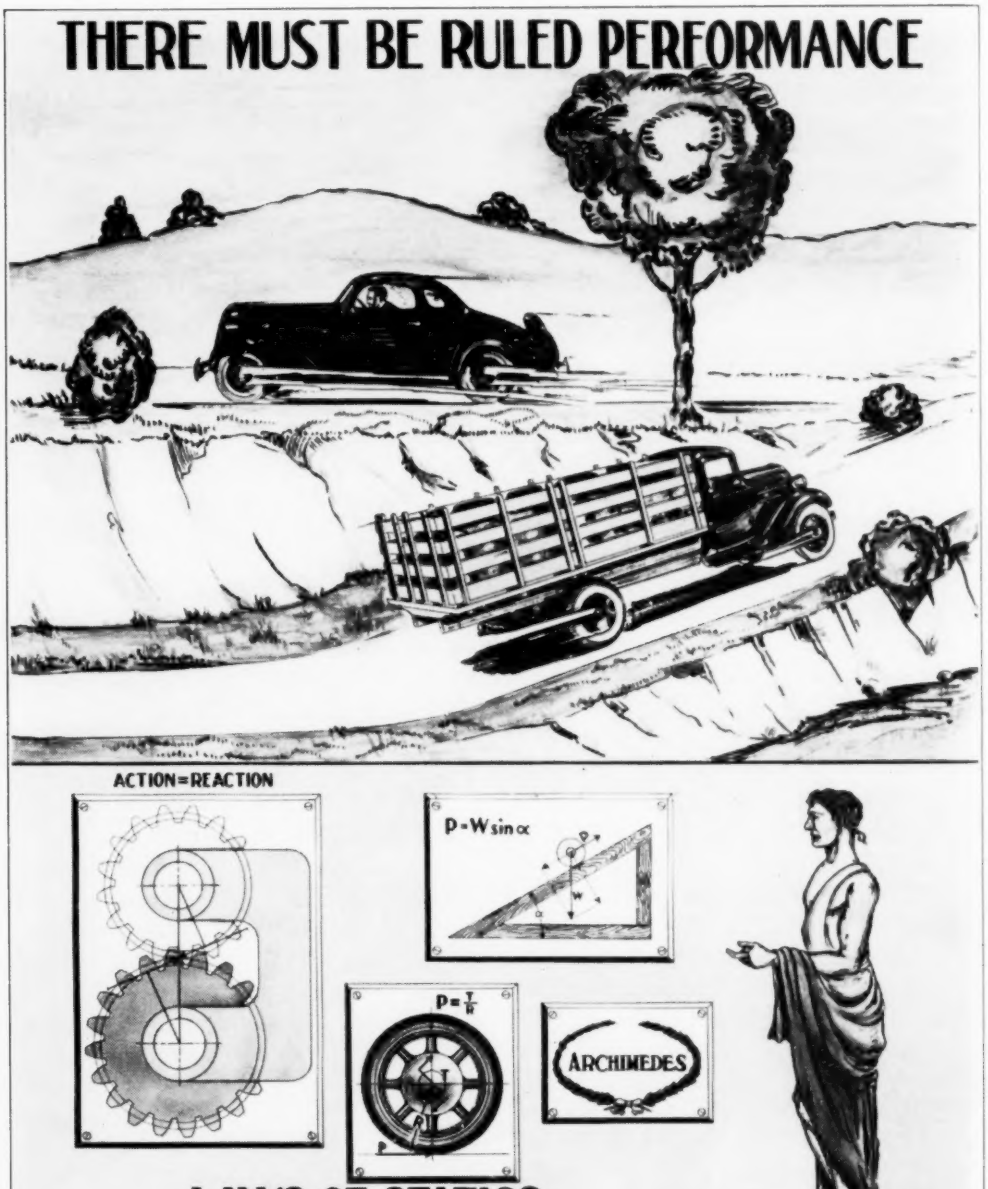
There Must Be Ruled Performance
There Must Be Balanced Power
There Must Be Measured Strength

Each of these requirements is qualified; softened and for good reasons. Restraint and compromise are needed.

Let us turn now to the sheet depicting **PERFORMANCE.** In the upper panel we see a landscape. A light coupé rushes over the level highway, while a heavily loaded truck laboriously climbs up the hill. The two vehicles perform quite differently and this difference of performance is inherent in them. Neither could do what the other does. Still they are using the same power plant. The difference is produced by the means interposed between the engine and the road. All these means fall

* The Chevrolet Motor Division is printing a large poster, suitable for class room use, on which will be reproduced the three posters illustrated in this article. In addition, this large poster will contain several other illustrations depicting the use of mathematics in motor car engineering. Copies of this poster, and a commentary concerning the illustration will be supplied, free of charge, to mathematics teachers and others to whom this material may be interesting. Address your requests to: Educational Service Department, Chevrolet Motor Division, General Motors Sales Corporation, General Motors Building, Detroit, Michigan to whom *The Mathematics Teacher* is indebted for this article.

under the classification of "simple machines" and so does the hill road itself. These machines evidently have their lever, toothed gears, wheel and axle, block and tackle, screw, inclined plane—make use of the **LAWS OF STATICS,** the



humble name because they are simple for everybody but the manufacturer and the road builder.

Since all simple machines—comprising lower panel of the sheet is devoted to this branch of mechanics which was first studied and understood by mankind in the glorious days of ancient classical

civilization. The symbolic figure of a man of these distant days presents four plaques, one of them commemorating the name of Archimedes, the great Greek mathematician, who was the founder of the science of statics. Of the three remaining plaques one represents a spur-gear drive. Such a drive is an excellent example of the universal law: *Action equals reaction.* Another plaque shows a driving road wheel whose action formula is given in the form $P = T/R$. Here T is the axle torque, R the rolling radius and P the tractive road force that propels the vehicle. The last plaque shows a rolling weight W pulled by a force P up an incline of angle α, it shows the typical parallelogram of forces and gives the equation $P = W \sin \alpha$.

Turning to the sheet assigned to **POWER,** we see spotlighted in the upper panel an automobile engine, well known to the American motorist, and in a smaller spotlight the crankshaft of the same engine. The engine is mounted on a test stand and the swirling fan blades betray that the engine is being tested, transmitting its power through the torque tube to a resistance outside of the spotlight.

If the headline of the sheet proclaims "Balanced Power," the term "balanced" is used both in its literal and its figurative meaning. The counterweights of the six-cylinder crankshaft in combination with this shaft's peculiar angular spacing of the piston drives, balance the moving masses against each other dynamically.

But this kind of dynamic mass balance has to be supplemented by a fine balance between the energy output of the engine and the power requirement of the vehicle, a balance that produces both a rich performance and a small fuel consumption, a balance furthermore that assures comfort for the driver and long life for the motor.

That means then that the engine has to be right both from the standpoint of **ENERGY** and of **DYNAMICS,** the two branches of physics to which the lower panel of this sheet is devoted.

Pallas Athena kindly presents a memorial tablet in honor of Newton and of Nicolas Carnot, together with five informative plaques of the two sciences fostered by these men.

Dynamics, as we know it today, is based directly upon the **LAWS OF MOTION** of the great Newton. Of course, in the meantime much has been added that makes dynamics a separate branch of mechanics so that it is conveniently applicable to engineering problems in which Newton was not and could not be interested.

One of the plaques gives the derivative expressions for uniformly accelerated straight line motion

$$v = \frac{ds}{dt} \qquad \text{and} \qquad a = \frac{dv}{dt}$$

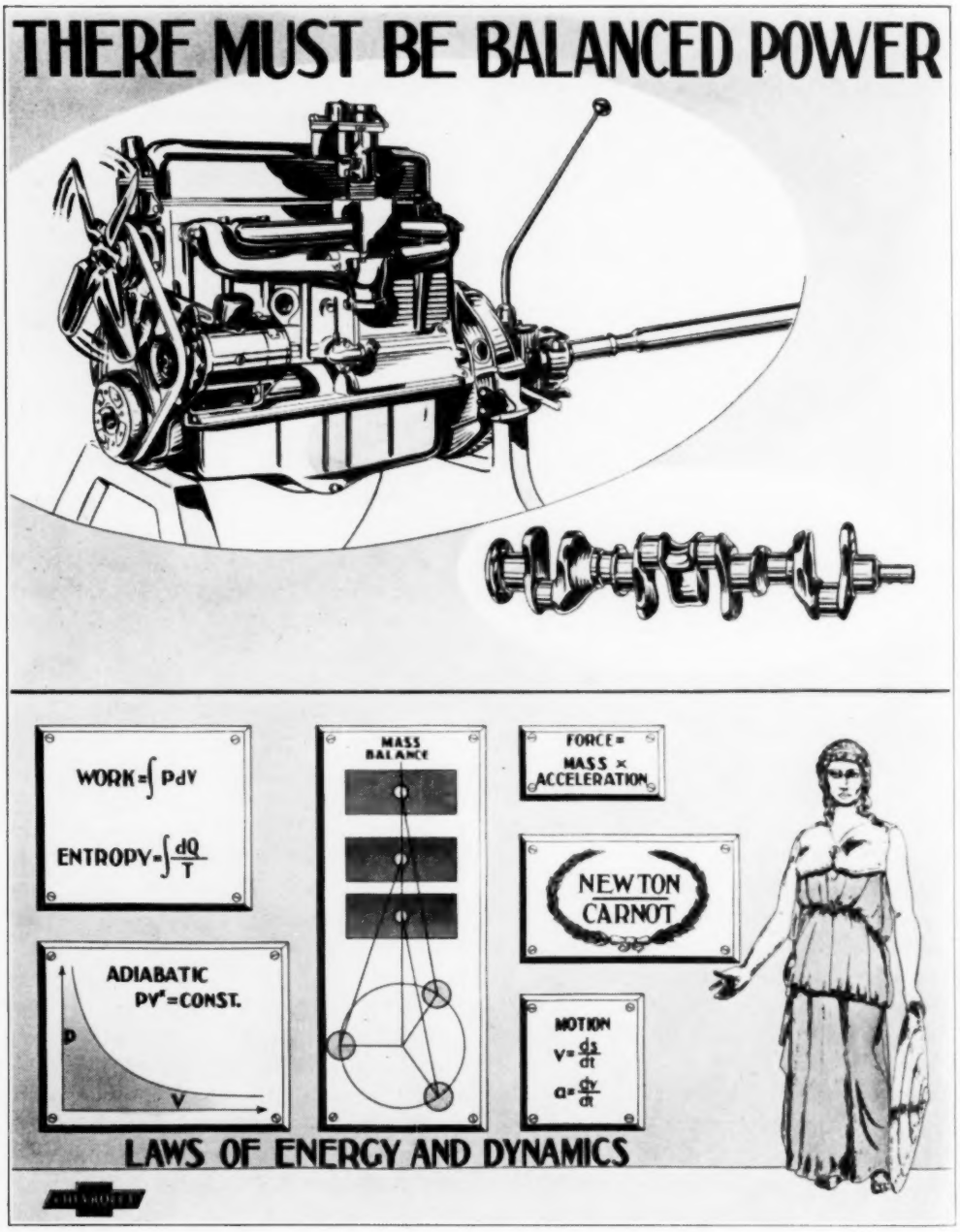
with v the velocity, s the space traveled, t the time and a the acceleration.

A second plaque states: Force = Mass × Acceleration. This suggests that force can be considered the effect of acceleration (or deceleration, if a is negative) as well as its cause.

The plaque in the center, entitled *Mass Balance* is a graphical illustration of the engineering possibilities of balancing the reciprocating parts of a motor. Notwithstanding their irregular position, the three reciprocating weights shown, are in perfect balance, if considered in one plane. This scheme is utilized in 6-cylinder line engines of the 4-stroke type, where the centers of gravity of three pairs of pistons are always in one plane.

Nicolas Carnot was chosen as the herald of the **LAWS OF ENERGY** for his epochal contributions to the science of thermodynamics, which is the mother of the modern combustion engine as well as of the present-day theories of the structure of the universe. It was the "Carnot Cycle" which revealed and explained to the world that the efficiency of a heat engine depends upon the difference of the temperatures between which the engine operates. The application of this principle

has led to a steady rise of compression in the development of internal combustion engines. Modern astronomy tells us that energy throughout the universe behaves in the same manner.



The two plaques at the left can give no more than a feeble suggestion of the mysterious laws of energy. The forms of energy found in an automobile—outside of electricity, which is only auxiliary—are heat and work, the engine transforming a part of the combustion heat into work and the chassis consuming this work by converting it, in the last analy-

sis, back into heat. The upper left plaque tries to convey the idea of this transformation of heat into mechanical energy. If work, as the tablet says, is the sum-product ($\int PdV$) of gas pressure (P) and volume increase (dV), heat must also be the product of two such quantities possessed by the gas. Of these two, one is sensible. It is the temperature which takes the place of pressure. The other quantity corresponding to the gas volume is not sensible and not measurable directly. It is, however, measurable mathematically and has been termed *Entropy* (meaning transformation). It is expressed by the equation

$$\text{Entropy} = \int_0^e \frac{dQ}{T}$$

where dQ refers to the heat quantities gradually received (or lost) by a pound of gas, while the latter changes from the zero state for entropy (32° F., 14.7 lb./sq. in.) to the end state "e," and where T refers to the corrresponding absolute temperatures ("absolute zero" is about −274° F.). The above expression is the renowned Integral of Clausius. This entropy integral—which is universally designated by the symbol S—now has for a specified end state "e" always the same value S_e, no matter under which intermediate pressure and temperature conditions the heat quantities dQ have been imparted and how great or small their sum has been. Thus entropy is as much a characteristic of a given gas state as temperature, pressure and specific volume. Two of these characteristics determine the two remaining ones. The heat Q now, imparted to a pound of gas between the states "1" and "2," is expressed by the equation

$$Q = \int_{s_1}^{s_2} T\,dS$$

dS referring to the changing entropies and T to the corresponding absolute temperatures.

The last remaining plaque of this sheet depicts the adiabatic expansion line of a gas, always wanted and never quite realized in motor engineering. This curve (with pressure plotted against volume) represents the transition—within a motor cylinder—of the initial compressed and burnt gas to its expanded state near the end of the working stroke, under the assumption that the gas during this transition has neither received nor lost heat through the cylinder and piston walls. The equation of this adiabatic expansion line is

$$PV^k = \text{Constant.}$$

Here P is the absolute pressure, V the specific volume and k the adiabatic exponent (For air $k = 1.4$).

The last of our three sheets is inscribed to **STRENGTH,** a most necessary property of a good car and difficult to obtain within the permissible weight range. In the upper panel we see the frame and the rear spring of a leading motor car. The dim outlines of a stone bridge appropriately are part of the decorative scheme of this sheet. Immediately we sense the difference of the strength embodied in the frame and that given to the spring. The object of a good frame design is a high degree of rigidity, while the essence of a spring is a high degree of flexibility.

To the physicist and engineer, strength, rigidity and flexibility are only three aspects of a single phenomenon, **ELASTICITY.** To this department of physics the lower panel of the sheet is devoted. An archer, whose weapon is the very symbol of elasticity, presents five plaques, one of them a memorial tablet in honor of Hooke, Euler and Hertz.

Robert Hooke, a contemporary of Newton, provided the basis of the study of elasticity by his statement that the extension of a truly elastic rod under a pull is proportional to the pulling force. This statement, "Hooke's Law," is given on the upper left plaque in its shortest and most comprehensive form:

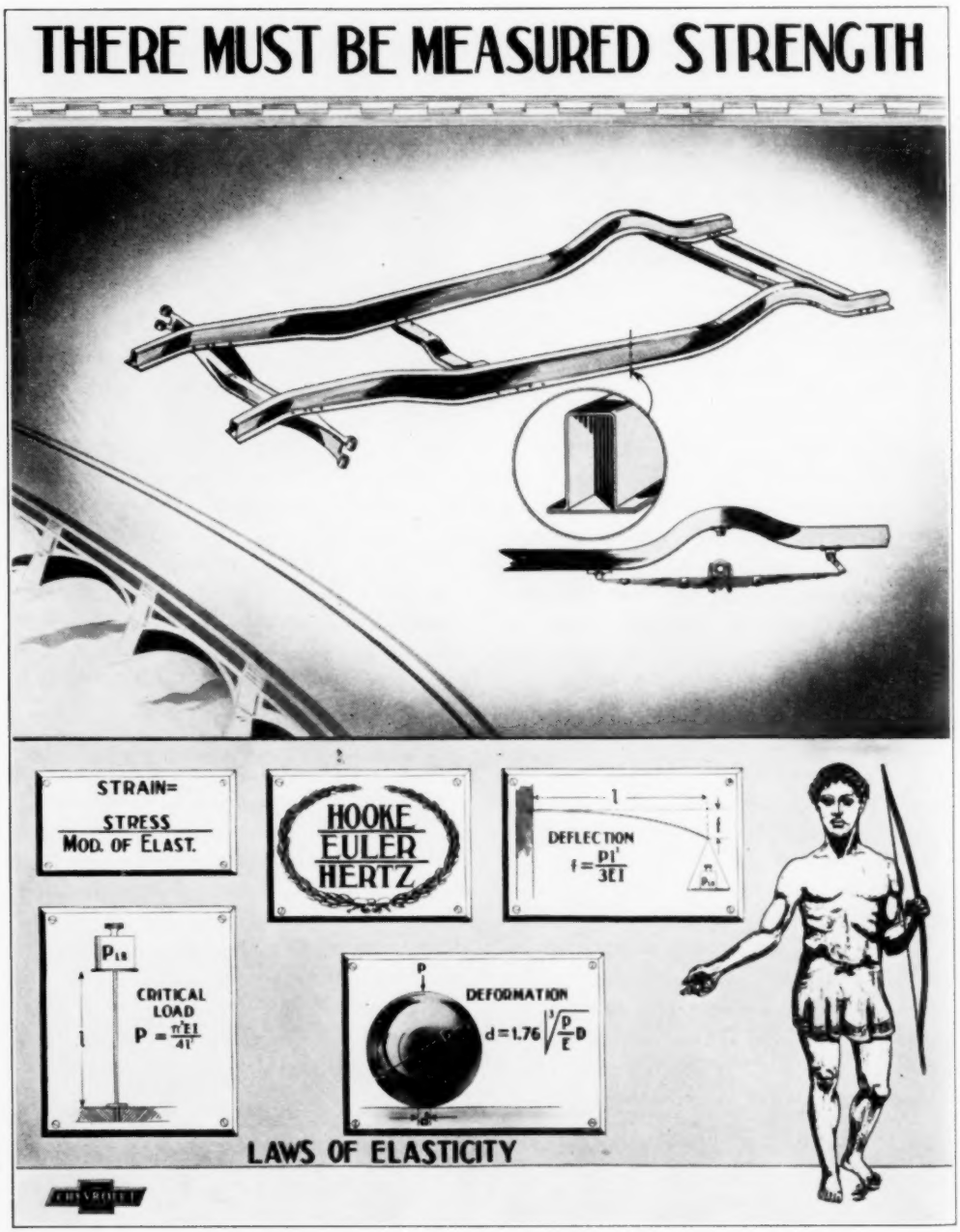
$$\text{Strain} = \frac{\text{Stress}}{\text{Modulus of Elasticity}}.$$

The great mathematician Leonard Euler has contributed to our knowledge of

elasticity by his famous equation stating the critical load for a slender column. This formula, with an illustration of the case, is given on the lower left plaque. It reads:

$$P = \frac{\pi^2 EI}{4l^2}$$

with E the modulus of elasticity, I the smallest equatorial moment of inertia, l the length of the column and P the critical load, and π approximately equal to 3.14159 . . . , is the ratio of the circumference of a circle to its diameter.



Heinrich Hertz, the great physicist and

discoverer of the Hertzian waves—used in wireless and radio communication—also has made a most important contribution to the science of elasticity. With purely mathematical means he found the contact stresses and deformations between bodies, having curved surfaces, when pressed together. These "Hertz equations" form the theoretical basis in the design of ball bearings and other antifriction bearings. The problem, depicted on the lower center plaque, of a steel ball being pressed against a steel plate, is solved by the simplest form of the Hertz equations, namely by:

$$d = 1.76\sqrt[3]{\frac{PD}{E}}$$

with d the diameter of the minute contact circle, P the pressing force, E the modulus of elasticity of steel and D the diameter of the ball.

The last remaining plaque illustrates the deflection of a cantilever formed by a horizontal rectangular spring leaf and loaded at the free end by a weight P. The vertical deflection of the free end is found by the equation

$$f = \frac{Pl^3}{3EI}$$

with f the deflection, P the weight, l the length of the cantilever, E the modulus of elasticity, and I the moment of inertia with reference to the neutral axis.

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The Importance of Certain Concepts and Laws of Logic for the Study and Teaching of Geometry*

By NATHAN LAZAR
Alexander Hamilton High School, Brooklyn, N. Y.

CHAPTER IV

AN EXTENSION OF THE CONCEPT AND OF THE LAW OF CONTRAPOSITION

AN EXTENSION OF THE DEFINITION OF A CONTRAPOSITIVE

IN THE transformations discussed in the foregoing chapters whether they were made on a theorem of the simplest type, such as

$$h \rightarrow c$$

or of the type

$$h_1h_2h_3 \cdots h_m \rightarrow c_1c_2c_3 \cdots c_n$$

the contrapositive was obtained by contradicting the *entire* hypothesis and the *entire* conclusion and interchanging their places. The difficulties resulting from the performance of such an operation upon a theorem of the latter type were pointed out. It is proposed here to extend the scope of the concept of contrapositive to include those transformations where the contradictory of the conclusion—if there is only one—is transposed to the hypothesis, while only one of the hypotheses is contradicted and transposed to the side of conclusion.

Thus the theorem

$$h_1h_2 \rightarrow c$$

which yielded only one contrapositive on the traditional definition, will now yield in addition the following two:

$$h_1\bar{c} \rightarrow \bar{h}_2$$
$$h_2\bar{c} \rightarrow \bar{h}_1$$

The first of the above contrapositives is to be interpreted as follows: if the first condition in the hypothesis is true and the conclusion is denied then the second condition of the hypothesis is false. The second contrapositive will be interpreted in a similar manner.

A theorem of the type

$$h_1h_2h_3 \rightarrow c$$

having three data in the hypothesis will yield the following contrapositives:

$$\bar{c}h_2h_3 \rightarrow \bar{h}_1$$
$$h_1\bar{c}h_3 \rightarrow \bar{h}_2$$
$$h_1h_2\bar{c} \rightarrow \bar{h}_3$$

Similarly a theorem having n conditions in the hypothesis and *one* conclusion will yield n contrapositives by interchanging the contradictory of *one* of the hypotheses and the contradictory of the conclusion.

An Extension of the Law of Contraposition

More important than the mere extension of the name *contrapositive* to the patterns of theorems enumerated above is the law (to be proved later) that the contrapositives derived from the same theorem are equivalent to each other and to the "mother" theorem as well, i.e., if the original theorem is true, each one of the contrapositives is true and conversely.

Illustration of the above contrapositives from geometry. Before attempting a general proof of the validity of the newly formed contrapositives it might be well to give a few illustrations taken from geometry.

To illustrate the type $h_1h_2 \rightarrow c$, let us take again the theorem,

If a line bisects the vertex angle of an

* This is the third and last installment of Dr. Lazar's thesis. The first appeared in the March issue and the second in the April issue of *The Mathematics Teacher*. The entire thesis (66 pages) *bound in cloth* may be obtained postpaid for $1 from *The Mathematics Teacher*, 525 W 120th St., New York, N.Y.

isosceles triangle, it is perpendicular to the opposite side.

Data	Conclusion
triangle ABC	AD is perpendicular to BC
1. $AB=AC$	
2. AD bisects angle A	

In accordance with the analysis made in the preceding column, this theorem yields the following contrapositives:

Contrapositive I, obtained by interchanging the contradictories of datum 2 and of the conclusion

Data	Conclusion
ABC is a triangle	AD does not bisect angle A
1. $AB=AC$	
2. AD is not perpendicular to BC	

Contrapositive II, obtained by interchanging the contradictories of datum 1 and of the conclusion

Data	Conclusion
ABC is a triangle	$AB\neq AC$
1. AD is not perpendicular to BC	
2. AD bisects angle A	

The reader can easily convince himself by the usual geometric methods that contrapositives I and II are both true.

Examples of the type $h_1h_2h_3\rightarrow c$. The following familiar theorem will help illustrate the possible transformations that may be effected by means of the extended Law of Contraposition:

Any point in the perpendicular bisector of a line is equidistant from the ends of the line.

Data	Conclusion
1. CD is perpendicular to AB	$PA=PB$
2. CD bisects AB	
3. P is any point in CD	

Contrapositive I

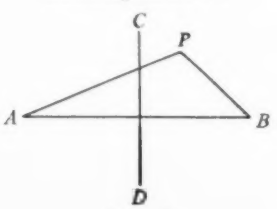

FIG. 4.

Data	Conclusion
1. CD is perpendicular to AB	P is not on CD
2. CD bisects AB	
3. $PA\neq PB$	

Contrapositive II

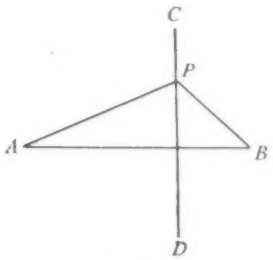

FIG. 5.

Data	Conclusion
1. CD is perpendicular to AB	CD does not bisect AB
2. P is a point on CD	
3. $PA\neq PB$	

Contrapositive III

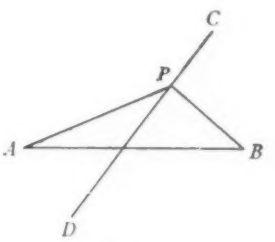

FIG. 6.

Data	Conclusion
1. CD bisects AB	CD is not perpendicular to AB
2. P is a point on CD	
3. $PA\neq PB$	

The following observations on the above contrapositives can easily be verified by the reader:

1. They are all true.

2. II and III can be proved directly, but the skill necessary for such proofs is far beyond the ability of the majority of students studying geometry.

3. Although they can all be proved by the indirect method such a method of proof is one that comes neither spontaneously nor easily to the average student of geometry.

4. It is of greater importance however to observe that if the claim for the validity of the Law of Contraposition can be substantiated, all the three statements can be asserted *without individual proof.* All that is necessary is to quote the original theorem and the Law of Contraposition.

Statement and proof of the extended Law of Contraposition. The law may be stated as follows: "If a theorem contains more than one hypothesis and only one conclusion, it is equivalent to any theorem formed from it by the interchange of the contradictory of one of the hypotheses with the contradictory of the conclusion."

The proof of the law follows, in the main, the proof given before on page 171. Although it is given here for the case of a theorem having three hypotheses, it does not lose any of its generality thereby.[1]

Any proposition of the pattern

$$h_1h_2h_3 \rightarrow c$$

may be translated into the following:

I. If the hypothesis h_1 *and* the hypothesis h_2 *and* the hypothesis h_3 are true, then the conclusion c is true.

The above statement in hypothetical form is evidently equivalent to the following statement in conjunctive form:

II. The assertion is false that h_1 is true and h_2 is true and h_3 is true and c is false.

But since rearranging the order of the constituent elements does not alter the meaning of a conjunctive statement, II is therefore equivalent to the statement:

III. The assertion is false that h_2 is true and h_3 is true and c is false and h_1 is true.

The above may now be transformed from its conjunctive form to the equivalent hypothetical form:

IV. If h_2 is true and h_3 is true, and c is false, then h_1 is false.

The above may be written in symbolic form,

$$h_2h_3\bar{c} \rightarrow \bar{h}_1$$

By similar reasoning it may also be shown that the following contrapositives are true:

$$h_1h_3\bar{c} \rightarrow \bar{h}_2$$
$$h_1h_2\bar{c} \rightarrow \bar{h}_3$$

It is of utmost importance to note that the extended Law of Contraposition does *not* apply to theorems having more than one conclusion. Theorems of the type

$$h_1h_2h_3 \rightarrow c_1c_2 \ldots$$

do not generally yield true theorems if one or more conclusions are interchanged with an equal number of hypotheses.

Advantages of the Law of Contraposition

Elimination of many troublesome indirect proofs. Every teacher is acquainted with the difficulties encountered by most students of geometry in the proof of the theorem,

Two lines are parallel if their alternate interior angles are equal.

The difficulties are due to several causes:

1. The figure used is often distorted, thus giving the pupil the impression that the two lines supposedly straight bend to meet in some point.[2]

2. The above difficulty can be removed by following Upton's advice and including two other diagrams, one for each of the assumptions contradicting the conclusion —one diagram in which the lines meet on the left, and another in which they meet on the right. However, even that device does not eliminate the more fundamental difficulty—the inability of the student to grasp the law of logic—that if the contradictory of a statement is false, or leads

[1] See, Lewis, C. I. *A Survey of Symbolic Logic*, p. 231, prop. 9.5. University of California Press. 1918. Lewis, C. I., and Langford, C. H. *Symbolic Logic*, p. 132, prop. 12.6. The Century Company. 1932. Both Hauber and Halsted seem to have known that certain types of theorems yield more than one contrapositive. They did not, however, analyze the conditions under which such transformations are possible.

[2] Upton, C. B., "The Use of Indirect Proof in Geometry and in Life." *The Fifth Yearbook of the National Council of Teachers of Mathematics*, pp. 119–121. Bureau of Publications. Teachers College. Columbia University. 1930.

to a false implication, the original statement must be true. Unfortunately there is little opportunity for preparing the ground by introducing that fundamental logical principle into the work preceding the crucial theorem. There is, moreover, little opportunity for utilizing the principle after the crucial theorem has been passed.

Note, however, that early in the course the Law of Contraposition can be introduced as an axiom of logic, even in connection with simple axioms and preliminary theorems. When the time for the presentation of the troublesome theorems arrives, usually in the fifth or sixth week of the term, the word *contrapositive* has taken on a definite meaning in the vocabulary of the students, and the Law of Contraposition has become part of their intellectual equipment. They find no difficulty in understanding the following approach:[3]

(a) The theorem

The exterior angle of a triangle is larger than either remote interior angle

is proved in the usual way.

(b) The following concepts are introduced: parallel, alternate interior angle, transversal.

(c) By means of the theorem in (a) the following corollary is proved without difficulty:

If two lines are not parallel, the alternate interior angles are not equal.

(d) Since the above corollary is true, its contrapositive is also true:

If the alternate interior angles are equal, the lines are parallel.[4]

Another theorem that is troublesome to many pupils in geometry for reasons similar to those given above, is

If two parallel lines are cut by a transversal the alternate interior angles are equal.

The following are some of the methods of proof commonly employed:

[3] See Halsted, *The Elements of Geometry*, p. 48. John Wiley and Sons. 1885.

[4] The above comments apply only to those textbooks that follow, in the main, Euclid's sequence.

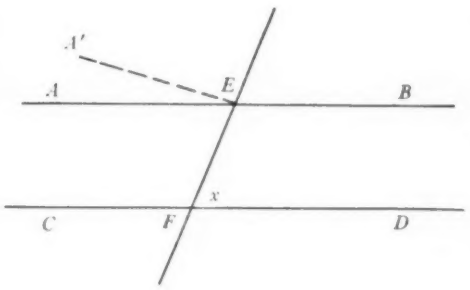

FIG. 7.

Data	Conclusion
1. AB is parallel to CD	angle x = angle AEF
2. A transversal intersects AB and CD at E and F respectively	

I. If angle $AEF \neq$ angle x it is either less than angle x or greater than angle x. If it is less than angle x, construct $A'E$ so that $A'EF = x$. $\therefore A'E$ is parallel to CD. Since AB is given parallel to CD, that would make two intersecting lines both parallel to a third line.

Since that is impossible, the assumption that angle AEF is less than angle x is false. The assumption that angle AEF is greater than angle x also leads to a false conclusion. Since the other two assumptions are untenable the only one left is that angle AEF = angle x.

II. Another proof, by the method of coincidence, is as follows: Construct $A'E$ making angle $A'EF$ = angle x. $A'E$ is therefore parallel to CD. But, by hypothesis, AE is parallel to CD. $A'E$ and AE must coincide, since through one point (E) one and only one line may be drawn parallel to a second line. Therefore angle $A'EF$ coincides with angle AEF. But angle $A'EF$ = angle x by construction. Therefore angle AEF = angle x.

III. A third method consists of the following two steps:

(A) Proof of the preliminary theorem

If a transversal to two lines makes a pair of alternate interior angles unequal, the lines are not parallel.

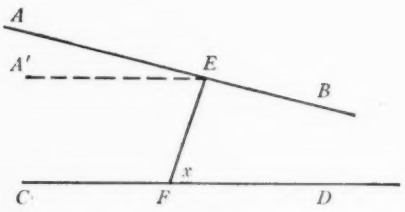

FIG. 8.

Data

AB and CD cut by a transversal EF so that

angle $AEF \neq$ angle x.

Conclusion

AB is not parallel to CD

Since angles AEF and x are not equal to each other, one of them is the larger one. Assume that angle AEF is the larger one. Construct angle $A'EF =$ angle x. $A'E$ must then be parallel to CD.

Since through a point, only one line can be drawn parallel to a second line, the other line AB, passing through E cannot be parallel to CD.

(B) Proof of the main theorem by the method of exhausting the alternatives. See figure 8. Either angle $AEF =$ angle x, or angle $AEF \neq$ angle x.

If angle $AEF \neq x$, then AB is not parallel to CD by III (A), but that contradicts the hypothesis. Therefore angle $AEF =$ angle x.

The Law of Contraposition affords a method which seems preferable to those enumerated above.

1. Prove III (A) in the usual manner. The proof is simple, direct, and offers no difficulty.

2. Transform III (A) into its contrapositive equivalent

If two lines are cut by a transversal, the alternate interior angles are equal.

By the Law of Contraposition the above theorem is true, since III (A) was proved true.

Halsted[5] seems to have overlooked the possibility of using the Law of Contraposition in the proof of this theorem and used in its place the clumsy method II, outlined above. He did use the law, however, to derive from the main theorem its contrapositive III (A).

A third theorem that is often proved by the method of coincidence is the following:

A line joining the midpoints of two sides of a triangle is parallel to the third side.

The difficulties encountered in the indirect methods of proof of the above theorem have been so thoroughly analyzed elsewhere that they will not be repeated here.[6] The following procedure is presented here more as another illustration of the power of the contrapositive method than as a procedure recommended for classroom use.[7]

I. The following theorem is proved in the usual way:

If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal.

II. The corollary stated below can now be deduced:

If a line is parallel to one side of a triangle, and bisects another, it bisects the third.

III. The inverse of the above corollary can now be shown to be true:

If a line bisects one side of a triangle, and is *not* parallel to a second, it does *not* bisect the third.

The proof of the above follows:

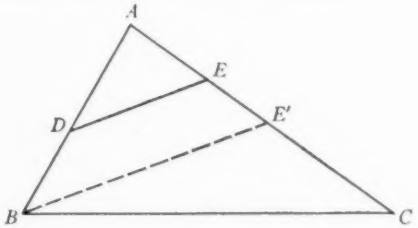

FIG. 9.

[5] See Halsted, *op. cit.*, p. 50.

[6] See Upton, *op. cit.*, pp. 120–121.

[7] Most likely the reader has seen the above theorem proved by the method of prolonging the line its own length and joining the end to the nearest end of the base. In some books it is postponed until the demonstration of the theorem: If a line divides two sides of a triangle proportionally, it is parallel to the third side.

Data

triangle ABC

1. D is the midpoint of AB
2. DE is not parallel to BC

Conclusion

E is not the midpoint of AC

Proof

Through B construct BE' parallel to DE. By the previous corollary II, $AE=EE'$. But $EE'\neq EC$, therefore $AE\neq EC$, and E cannot be the midpoint of AC.[8]

IV. After the proof of the previous theorem has been established, the truth of its contrapositive follows without further demonstration.

Data	Conclusion
triangle ABC	DE is parallel to BC
1. D is the midpoint of AB	
2. E is the midpoint of AC	

The verbal equivalent of the above analysis is:

If a line joins the midpoints of two sides of a triangle it is parallel to the third.[9]

A fourth theorem that is invariably proved by some form of the method of coincidence is the following:

If a line divides two sides of a triangle into segments that are proportional it is parallel to the third side.

No teacher of geometry need be reminded of the bewildered expression on the faces of the pupils when they are told that the two lines, one of which seems to be a part of the other, really is equal to it, and that the other two lines which undoubtedly intersect, must by some magic get together and coincide.

The method of contraposition affords another approach to the proof of the theorem.

I. Proof, by the usual method, of the fundamental theorem:

If a line is parallel to one side of a triangle, it divides the other two sides into segments that are proportional.

[8] The above proof is also valid in the case where BE intersects the prolongation of AC.

[9] Halsted seems to have overlooked here, too, the possibility of utilizing the Law of Contraposition. See, *op. cit.*, p. 74.

II. Proof, by the method below, of the inverse (opposite) of the above theorem: If a line intersecting two sides of a triangle is not parallel to the third, the segments are not in proportion.

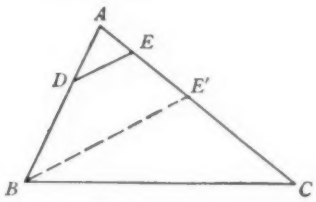

FIG. 10.

Data

triangle ABC

DE is not parallel to BC

Conclusion

$AD:DB\neq AE:EC$

Proof

Construct BE' parallel to DE.[10]
By Theorem I, $AD:DB=AE:EE'$
Since $EE'\neq EC$,
therefore, $AE:EC\neq AE:EE'$,
and $AD:DB\neq AE:EC$.

Note that in the fourth step the assumption is made that if two fractions have the same numerators but unequal denominators they are unequal.

III. Once the truth of the above is established its contrapositive is also true:

If a line divides two sides of a triangle into proportional segments, it is parallel to the third side.[11]

Independent Discovery of Theorems

Important as is the application of the Law of Contraposition to the simplification of the above indirect proofs, its real value lies in the power that it gives every student of mathematics (and in particular of geometry) to use every true proposition as a starting point for the discovery of more

[10] See note 8 in the preceding column.

[11] Halsted did not use the method of contraposition to prove the theorem under discussion. He proved it by quoting the following theorem: A given sect can be divided internally into two segments having the same ratio as any two given sects, and also externally unless the ratio be of equality; and in each case there is only one such point of division.

true theorems. A few illustrations will make this claim clearer.

1. The following is the analysis of the axiom:

If equal quantities are added to equal quantities, their sums are equal.

If $a=a'$
and $b=b'$

then $a+b=a'+b'$

Contraposing the above we get

if $a+b\neq a'+b'$
and $b=b'$

then $a\neq a'$

which may be expressed as follows:

If equals are subtracted from unequals, the remainders are unequal.

2. The simple theorem,

If two adjacent angles have their exterior sides in a straight line, they are supplementary,

yields the important contrapositive:

If two adjacent angles are not supplementary, their exterior sides are not in a straight line.

3. Even complicated theorems yield contrapositives very easily. The theorem,

If two sides and the included angle of a triangle are equal respectively to two sides and the included angle of the other, the third side of the first is equal to the third side of the second,

becomes in its contrapositive form:

If two triangles have two sides of one equal respectively to two sides of the other, and the third side of the first is not equal to the third side of the second, the angle opposite the third side of the first is not equal to the angle opposite the third side of the second.

The logical transformation does not by itself indicate the exact nature of the inequality; but the little knowledge gained, almost without effort, has the value of starting the student of mathematics on a road of independent discovery.

There is no point in enumerating any more theorems discoverable by the law of logic expounded here; for every theorem in mathematics (and incidentally in other fields of human knowledge) gives rise to numerous meaningful and true contrapositives. Very often they do not resemble at all the original theorem from which they spring unless one is trained specifically to be on the lookout for such equivalences.

Contrapositives Are Logically Alike But Psychologically Unlike

There is a constant danger of minimizing the importance of contrapositive statements because of their logical equivalence to the original theorem. A careful consideration of the preceding pages may convince the skeptic that the diagrams of the contrapositives and the facts they communicate are essentially different in meaning from those of the original statement. Moreover, the fact that for more than two thousand years many theorems in Euclid that are logically similar escaped detection by the keenest of the world's thinkers may lead to the following conclusions:

1. There is a general tendency to consider only those statements equivalent which convey the same information in recognizably similar forms of expression.

2. Logical laws and patterns must be made the focus of attention and a subject for instruction if they are to function effectively in dealing with any subject matter

The Law of Contraposition as a Method of Research

So far, it was pointed out that the Law of Contraposition can be used as a means for discovering new theorems when the hypotheses and the conclusion of a certain theorem are already known. Often it may be used as a method of finding out whether a certain set of hypotheses will yield any conclusion and, if it does, what the conclusion or conclusions will be.

1. Let us take the following problem:

Data	Conclusion
two triangles ABC and $A'B'C'$	?
1. $a=a'$	
2. $b=b'$	
3. $A\neq A'$	

What conclusion, if any, can be drawn from the above data?[12]

Regardless of the method used to arrive at the answer, it most likely consisted of a mixture of intelligent guessing guided by experiment, and by what is often called, for the lack of a better term, geometric intuition. Note, however, the directness with which the problem can be solved by logic. One of the contrapositives of the theorem can be obtained by contradicting the conclusion and datum 3, and exchanging their places. Leaving datum 3 in the contrapositive blank, the following results:

Data	Conclusion
two triangles ABC and $A'B'C'$	$A=A'$
1. $a=a'$	
2. $b=b'$	
3. ?	

The problem now becomes: What datum should be inserted in 3 so that it may yield, in combination with the other two, the conclusion $A=A'$? The answer to the question is obviously either $c=c'$, (using the theorem *s.s.s.* = *s.s.s.*) or $C=C'$, (using the theorem *s.a.s.* = *s.a.s.*). Since the required datum is either $c=c'$ or $C=C'$, the required conclusion in the original problem is $c\neq c'$, or $C\neq C'$.

2. The reader can easily convince himself that the method discussed above has a wide range of applications. One more illustration will, however, be given here:

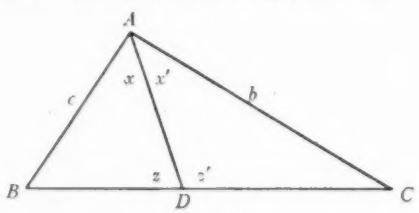

FIG. 11.

Data	Conclusion
triangle ABC	?
1. $c\neq b$	
2. $x=x'$	

[12] The reader is advised to work out the answer to the above question before proceeding to the analysis presented in this study.

Out of his fund of knowledge the mature student can arrive at various conclusions. Let us however see how the method recommended here would be used by a tyro. Transforming the above into its contrapositive form, the following is obtained:

Data	Conclusion
triangle ABC	$c=b$
1. ?	
2. $x=x'$	

It is evident that one datum that would, in combination with datum 2, yield $c=b$, is $z=z'$, (using *a.s.a.* = *a.s.a.*). Since $z=z'$ is the missing datum, $z\neq z'$ is the conclusion to the original theorem. But since $z\neq z'$ is equivalent to the statement, AD is not perpendicular to BC, the student is thus led to discover:

If two sides of a triangle are not equal, the bisector of the included angle is not perpendicular to the third side.

Although the technique described in this section has interesting, and perhaps important, ramifications, no claim is made that it is an "open sesame" to great discoveries in mathematics or a panacea for all pedagogic ills.

Application of the Law of Contraposition to Loci Problems

Methods of Proving a Locus

It is a commonplace of mathematical reasoning that in order to establish the fact that a certain locus fulfills certain given conditions, it is necessary to prove *two* propositions:

1. That any point on the prescribed locus satisfies the given condition.

2(a). That any point which satisfies the given condition is on that locus.

Many textbooks state categorically, without explanation or proof that in place of the converse, 2(a), the following inverse may be demonstrated:

2(b). Any point not on the curve does not satisfy the given condition.

The early introduction of the Law of Contraposition would not only obviate the

necessity of including the proof but would also clarify the relation between the converse and the inverse propositions. Moreover it would also render unnecessary the inexcusable "proofs" by example given in some manuals for teachers or the indirect proofs found in others.[13]

Some books point out not only the equivalence of a converse of a theorem and its opposite but also the equivalence between a theorem and the converse of its opposite. Yet they overlook the important fact that the relation between the converse of a theorem and its opposite is exactly the same—contrapositive—as the relation between a theorem and the converse of its opposite. Thus in the section on *Relation between Converse, Opposite, and Converse of the Opposite*, Schultze states:

If the theorem is true, then the converse of the opposite is true.

If the opposite is true, then the converse is true.

If the converse is true, the opposite is true.

If the converse of the opposite is true, the theorem is true. These connections are easily proved by the indirect method.[13]

These connections are all examples of the same relation of contraposition. Moreover they do not have to be proved anew in each case, once the law is established.

Since the equivalences enumerated above are true, it is often pointed out that there are four ways of proving a locus problem.[14] Thus instead of the traditional method of proving

1. A theorem and its converse,

one may prove

2. A theorem and its opposite (inverse),

or

3. The converse of a theorem and the converse of the opposite (contrapositive),

or

4. The opposite (inverse) of a theorem and the converse of the opposite (contrapositive).

[13] Schultze, A., *The Teaching of Mathematics in Secondary Schools*, p. 145.
Christofferson, H. C., *Geometry Professionalized for Teachers*, p. 131.

[14] Schultze, *op. cit.*, pp. 145, 219.
Christofferson, *op. cit.*, pp. 132–133.

Nine Ways of Proving a Locus Problem of the Type $h_1h_2 \rightarrow c$

If the thesis presented here is true, then the above four methods of establishing the validity of a locus comprise the minimum number of possible ways of attacking a problem of that type.[15] The specific number of ways will depend upon the number of conditions in the hypothesis. Thus if a theorem has two data in the hypothesis and one conclusion, it will yield the following:

1. Two converses, by interchanging one condition in the hypothesis with the conclusion,

2. Two inverses, by contradicting one condition in the hypothesis and the conclusion, and

3. Two contrapositives, by contradicting one hypothesis and the conclusion and interchanging their places.

Schematically the following will result:

I. Theorem	II. Contrapositives
$h_1h_2 \rightarrow c$	(a) $h_1\bar{c} \rightarrow \bar{h}_2$
	(b) $h_2\bar{c} \rightarrow \bar{h}_1$
III. Converses	**IV. Inverses**
(a) $h_1c \rightarrow h_2$	(a) $h_1\bar{h}_2 \rightarrow \bar{c}$
(b) $h_2c \rightarrow h_1$	(b) $\bar{h}_1h_2 \rightarrow \bar{c}$

The following points should be noted:

1. Theorem I is equivalent to each one of the two contrapositives in II, which are also equivalent to each other.

[15] The writer has been unable to find any locus problem that has only one datum in the hypothesis.

The following analysis will attempt to show that, by its very nature, a locus problem must have at least two conditions in its hypothesis: The simplest way of establishing that a curve C is the locus of a point fulfilling certain conditions is to show (1) that any point P on the curve fulfills the given condition, and (2) that any point P not on the curve C does not fulfill those conditions. But the theorems to be proved contain at least the following *two* hypotheses:

(a) a curve C, fulfilling certain conditions, and

(b) a point P that is on C, (or is not on C).

2. Converse III (a) is equivalent to Inverse IV (a).
Converse III (b) is equivalent to Inverse IV (b).

3. Converse III (a) is not equivalent to Converse III (b); nor is Inverse IV (a) equivalent to Inverse IV (b). Often one of the converses is true while the other is false; similarly with the inverses.

4. When establishing a locus, it is important to determine which one of the possible converses is asserted and to be proved.

5. If Converse III (a) is to be established one may either prove it or its equivalent contrapositive—Inverse IV (a)—or its other contrapositive (not enumerated above) $c\bar{h}_2 \rightarrow \bar{h}_1$.

6. In summary, one may prove one theorem from Column A and one theorem from Column B:

A	B
$h_1h_2 \rightarrow c$	$h_1c \rightarrow h_2$
$h_1\bar{c} \rightarrow \bar{h}_2$	$h_1\bar{h}_2 \rightarrow \bar{c}$
$\bar{c}h_2 \rightarrow \bar{h}_1$	$c\bar{h}_2 \rightarrow \bar{h}_1$

There are, therefore, *nine* methods of proving a locus problem which contains two conditions in the hypothesis.

Sixteen Ways of Proving a Locus Problem of the Type $h_1h_2h_3 \rightarrow c$

Similarly, when a theorem dealing with loci contains three data in its hypothesis, it will contain three data in the hypothesis of any one of the converses. The original theorem with its three contrapositives will yield four propositions, and any converse with its three contrapositives will also yield four propositions. The number of ways of proving a locus problem of that type would therefore amount to sixteen.

To illustrate: Let us take a locus problem which requires for its solution the proof of the theorem

$$h_1h_2h_3 \rightarrow c$$

and one of its converses, $h_1h_2c \rightarrow h_3$.

The theorem together with its contrapositives afford the four possibilities in Column D while the converse and its contrapositives yield the four in Column E:

D	E
$h_1h_2h_3 \rightarrow c$	$h_1h_2c \rightarrow h_3$
$h_1h_2\bar{c} \rightarrow \bar{h}_3$	$h_1h_2\bar{h}_3 \rightarrow \bar{c}$
$h_1\bar{c}h_3 \rightarrow \bar{h}_2$	$h_1c\bar{h}_3 \rightarrow \bar{h}_2$
$\bar{c}h_2h_3 \rightarrow \bar{h}_1$	$ch_2\bar{h}_3 \rightarrow \bar{h}_1$

Since the proof of a problem in locus requires the demonstration of one theorem from Column D and of one from Column E, there are therefore 16 ways of proving such problems.

Importance of the Multiplicity of Ways of Proving Locus Problems

The above facts undoubtedly have for the true mathematician a certain intrinsic interest that needs no other justification than "knowledge for its own sake." Even the student in the secondary school will be interested in knowing that there are many ways of solving a certain problem as long as he is not asked to know more than one or two.

It should be remembered, however, that in some cases it is easier to prove the inverse of a theorem than its converse, whereas in other cases the reverse is true. Without the inclusion of the Law of Contraposition as an integral part of the geometric curriculum the equivalence of those two transformations would rarely be utilized.[16]

Summary

In this chapter the traditional concept of the contrapositive is extended to include those transformations in which the contradictory of *only* one of the hypotheses is interchanged with the contradictory of the conclusion. A generalization of the Law of Contraposition is made (for the first time, it is believed) and its applications to geometry are indicated.

[16] See Schultze, *op. cit.*, pp. 219–220, for illustrative examples. Altshiller-Court, N., *College Geometry*, pp. 10–15. Johnson Publishing Co. 1925, contains the proofs of many locus problems. Some are proved by means of the theorem and its converse, others by the theorem and its inverse. No mention is made of the equivalence of converse and inverse.

Chapter V

LAW OF CONVERSES

In most textbooks in geometry, the following six theorems are proved independently of each other:[1]

1. If $a=b$, then $A=B$.
2. If $a>b$, then $A>B$.
3. If $a<b$, then $A<B$.
4. If $A=B$, then $a=b$.
5. If $A>B$, then $a>b$.
6. If $A<B$, then $a<b$.

Theorems 2 and 3 are usually combined into one proposition; so are 5 and 6. Theorems 4, 5, and 6 are, of course, the the converses, respectively, of 1, 2, and 3.

It has often been pointed out[2] that if the three theorems—1, 2, and 3—have been proved, their converses—4, 5, and 6—need not be proved independently but may be derived by logic from them.

Thus the following is the proof for Converse 4,

If $A=B$, then $a=b$.

If the conclusion $a=b$, is false, then either $a>b$, or $a<b$. If $a>b$, then, according to 2, $A>B$. But that is false since it denies the assumption of 4, that $A=B$. Similarly if the other alternative is entertained, that $a<b$, then according to 3, $A<B$. But that too contradicts the hypothesis of 4. Since the alternatives to the conclusion lead to a contradiction, the conclusion that $a=b$ must be true.

The remaining converses—5 and 6—may be proved in a similar manner.

History of the Law of Converses

It is interesting to note that whereas many textbooks in geometry use the above indirect method to prove converses 5 and 6, none, to the knowledge of the writer, used it to prove converse 4. Before proceeding to show how the Law of Converses is really an application of the Law of Contraposition, the following historical sidelight should be of interest.

Common Belief That De Morgan First Discovered the Law of Converses

The Law of Converses first appeared in English in De Morgan's works.[3] It has been implicitly assumed by subsequent writers that he was also the discoverer of that law. Thus Hirst,[4] a pupil of De Morgan, believed that his master was the first to discover the contrapositive redundances in Euclid.

Heath,[5] in his monumental work on Euclid, also seems to attribute the above law to De Morgan.

Contemporary writers relying upon the authority of Heath also claimed De Morgan as the original discoverer.

Hauber Announced the Law in 1829

We reproduce (on pp. 227–228) the title page and §287 of Hauber's book to prove beyond a shadow of doubt that Hauber anticipated De Morgan's publication of the Law of Converses by at least eleven years. Many remarks similar to those made by De Morgan on contrapositive

[1] The customary symbols are used below: A for angle A, a is the side opposite angle A, etc.

[2] Hauber, F. C., *Scholae Logico-Mathematicae*, p. 267. Heath, T. L., *The Thirteen Books of Euclid's Elements*, vol. I, pp. 284–285. Christofferson, H. C., *Geometry Professionalized for Teachers*, p. 130. Schultze, A., *The Teaching of Mathematics in Secondary Schools*, p. 146. Upton, C. B., "The Use of Indirect Proof in Geometry and in Life." *Fifth Yearbook of the National Council of Teachers of Mathematics*, p. 117. Smith, D. E., *The Teaching of Geometry*, p. 191. Beman, W. W. and Smith, D. E., *New Plane and Solid Geometry*, p. 34. McMahon, J., *Elementary Geometry*, pp. 48–49. Halsted, G. B., *The Elements of Geometry*, p. 7. Smith, D. E., *The Teaching of Elementary Mathematics*, pp. 278–279.

[3] De Morgan, A. (a) *First Notions of Logic*, p. 29. Second Edition. Taylor and Walton. 1840. (b) *Formal Logic*, p. 25. Taylor and Walton. 1847.

[4] Association for the Improvement of Geometrical Teaching, *Second Annual Report*, p. 15, 1872. See quotation on page 166. But see *Mind*, Old Series, Vol. 1, (1876), p. 425.

[5] Heath, *op. cit.*, vol. I, p. 256.

SCHOLAE

LOGICO-MATHEMATICÆ,

in quibus

ARS COGITANDI ET ELOQUENDI,

INVENIENDI ET DEMONSTRANDI

circa unam propositionem,

quae est

Euclidis Elementorum theorema primum,

multis modis et magna exemplorum varietate exercetur.

Proponuntur et

VARIA GENERALIA DE METHODO,

et nova quaedam

tum ad logicam theoreticam pertinentia, tum de porismatibus in analysi geometrica antiquorum.

Auctore

FRIDERICO CAROLO HAUBER,

Ephoro seminarii Maulbronnensis, Würtembergiae.

Cum tabulis lithographicis octo.

Reutlingae,

in officina literaria.

MDCCCXXIX.

equivalence in Euclid may also be found in Hauber, Drobisch and Matzka.[6]

The establishment of priority of discovery does not, of course, preclude the possibility of independent discovery of the law by De Morgan.

§ 287.

Quod generatim sic comprehendemus: Si genus aliquod dividatur in suas species duplici ratione, et singulis speciebus unius divisionis respondeant singulae species alterius ut attributa: vicissim etiam singulis speciebus alterius divisionis singulae species prioris ut attributa respondebunt.

Ut si genus quoddam A dividatur primum in species b, c, ac deinde in species β, γ: ut Omne A sit aut b aut c, et rursus Omne A sit aut β aut γ; et praeterea, quae sint ex specie b, iis attribuatur β; quae ex specie c, iis γ: his igitur positis, vicissim, quae sunt ex specie β, iis attribuetur b; et quae ex specie γ, iis attribuetur c.

Quod sic ostendetur, Si non iis A, quae sunt ex specie β, omnibus attribuatur b: cuicumque eorum non attribuitur, ei attribuetur c; quoniam Omne A est alterutrum, aut b aut c, per primam divisionem. Sed omnibus c etiam γ attribuitur per hypothesin. Ergo et omni β, quod non est b, attribuitur γ. Hoc vero fieri nequit: nulli enim β attribuitur γ, quia per hyp. nullum A est et β et γ, sed alterutrum eorum, per divisionem secundam. Non ergo iis A, quae sunt ex specie β, non omnibus attribuitur b: ergo omnibus attribuitur. Similiterque ostendetur, omnibus, quae sunt ex specie γ, etiam c attribui. Quod erat ostendendum.

Pone pro A triangulum, pro b aequicrus, pro c habens latus alterum altero majus; pro β, habens angulos lateribus subtensos aequales; pro γ, habens angulorum lateribus subtensorum alterum altero majorem. Et his substitutis habebis ipsam demonstrationem § 285. traditam.

[6] Hauber, *op. cit.*, ch. 7. Drobisch, M. W., *Neue Darstellung der Logik*, Leipzig: Leopold Voss. First Edition, 1836, pp. 162–163. Fifth Edition, 1887, pp. 234–239.

Matzka, W. "Betrachtungen einiger Gegenstände der Logic, mit besonderer Rücksicht auf ihre Anwendung in der Mathematik." *Archiv der Mathematik und Physik* (J. A. Grunert, Editor), Series I, vol. 6, (1845), pp. 353–369.

Proof of the Law of Converses

In the last section but one the Law of Converses was proved in a specific case. It may, however, be generalized in the following way:

I. Let P_1, P_2, and P_3 be a set of propositions such that they exhaust all possibilities among them, as well as being contraries of each other. Examples of such sets of propositions are:

A. $x = y$
   $x < y$
   $x > y$

B. a is younger than b
   a is older than b
   a is of the same age as b

C. This string has the same pitch as the other string.
   This string has a lower pitch than the other one.
   This string has a higher pitch than the other one.

The three statements in each set exhaust all possibilities with reference to the relation considered. Moreover one of the three must be true, and no more than one can be true.

II. Let Q_1, Q_2, and Q_3 be another set of propositions related to each other in the same way as P_1, P_2, and P_3.

III. Let it be assumed that

1. $P_1 \rightarrow Q_1$
2. $P_2 \rightarrow Q_2$
3. $P_3 \rightarrow Q_3$

According to the Law of Converses the following three converses of the above theorems are also true:

4. $Q_1 \rightarrow P_1$
5. $Q_2 \rightarrow P_2$
6. $Q_3 \rightarrow P_3$

The proof, using symbols, is as follows:

Since $P_1 \rightarrow Q_1$
and $P_2 \rightarrow Q_2$,
by *logical* addition: $P_1 + P_2 \rightarrow Q_1 + Q_2$,
by the Law of Contraposition: $(\overline{Q_1 + Q_2}) \rightarrow (\overline{P_1 + P_2})$,
by De Morgan's theorem: $\bar{Q}_1\bar{Q}_2 \rightarrow \overline{P}_1\overline{P}_2$.
Since, by hypothesis, Q_1, Q_2, and Q_3 exhaust all possibilities, $\bar{Q}_1\bar{Q}_2 \equiv Q_3$.
Similarly, $\overline{P}_1\overline{P}_2 \equiv P_3$.
By substitution, $Q_3 \rightarrow P_3$.

In words, the above proof may be put as follows:
Since P_1 implies Q_1,
and P_2 implies Q_2,
then, if either P_1 or P_2 is true, either Q_1 or Q_2 is true.
Writing the above in contrapositive form:
The denial of "either Q_1 or Q_2 is true" implies the denial of "either P_1 or P_2 is true."
But, according to De Morgan's theorem, the denial of "either Q_1 or Q_2 is true" is "Both Q_1 and Q_2 are false," and the denial of "either P_1 or P_2 is true" is: "Both P_1 and P_2 are false."
Therefore, the above contrapositive becomes the following:
The statement that "both Q_1 and Q_2 are false" implies the statement that "both P_1 and P_2 are false."
Since by hypothesis Q_1, Q_2, and Q_3 are propositions so related that one of them must be true, the statement that "both Q_1 and Q_2 are false" is equivalent to the statement that "Q_3 is true." Similarly, the statement that "P_1 and P_2 are false" is equivalent to the statement that "P_3 is true."

By substitution in the above we obtain Converse 6, $Q_3 \rightarrow P_3$.

Similarly, the truth of Converses 4 and 5 may be established.

Applications of the Law of Converses to Geometry

In addition to the application given on page 226, the following should be mentioned:[7]

1. If c and c' are two chords in the same circle or in equal circles and d and d' are the respective distances from the center of the circle, the following theorems are true:

If $c = c'$, then $d = d'$
If $c > c'$, then $d < d'$
If $c < c'$, then $d > d'$

By the Law of Converses, the converses are also true.

2. Let c and c' be two chords in the same circle or in equal circles, and a and a' their respective minor arcs. The following theorems and their converses are true:

If $c = c'$, then $a = a'$
If $c > c'$, then $a > a'$
If $c < c'$, then $a < a'$

3. The above theorems and their converses are also true if c and c' represent two central angles in the same circle or in equal circles, and a and a' represent their corresponding arcs.

4. If A, B, C represent the angles of a triangle and a, b, c the opposite sides, the following set of theorems and their converses are true:

If $C = 90$, then $c^2 = a^2 + b^2$
If $C < 90$, then $c^2 < a^2 + b^2$
If $C > 90$, then $c^2 > a^2 + b^2$

Generalization of the Law of Converses

Limitation of the Above Law of Converses

The reader has undoubtedly noticed that both in the proof and in the applications of the Law of Converses, the hypothesis of each theorem contained only one statement. It is natural to wonder whether or not it is possible to prove that a law similar to the above will apply to a set of theorems of the following type

1. $MP_1 \rightarrow Q_1$
2. $MP_2 \rightarrow Q_2$
3. $MP_3 \rightarrow Q_3$

where P_1, P_2, and P_3, and Q_1, Q_2, and Q_3 obey the same restrictions as above, but the condition M is the same in all. It is unnecessary to point out that although, by hypothesis, P_1, P_2, and P_3 exhaust all possibilities, the conjunct propositions MP_1, MP_2, and MP_3 need not exhaust all possi-

[7] See Upton, *op. cit.*, pp. 118–119.

bilities. These hypotheses cannot, therefore, be operated upon in the same way as P_1, P_2, and P_3 were, in the proof on page 54.

Proof of the Generalized Law of Converses

The following extended Law of Converses will now be proved:

If
1. $MP_1 \rightarrow Q_1$
2. $MP_2 \rightarrow Q_2$
3. $MP_3 \rightarrow Q_3$

then the following converses are also true:

4. $MQ_1 \rightarrow P_1$
5. $MQ_2 \rightarrow P_2$
6. $MQ_3 \rightarrow P_3$

Proof:

Since $MP_1 \rightarrow Q_1$
and $MP_2 \rightarrow Q_2$

by logical addition: $MP_1 + MP_2 \rightarrow Q_1 + Q_2$ which is equivalent to, $M(P_1 + P_2) \rightarrow Q_1 + Q_2$. By the Law of Contraposition, the above may be transformed into

$$M\overline{(Q_1+Q_2)} \rightarrow \overline{(P_1+P_2)}$$

By De Morgan's theorem, the above becomes

$$M\bar{Q}_1\bar{Q}_2 \rightarrow \overline{P}_1\overline{P}_2 \qquad (I)$$

Since by hypothesis, $\bar{Q}_1\bar{Q}_2 \equiv Q_3$

and, $\overline{P}_1\overline{P}_2 \equiv P_3$

substituting in (I) the following is obtained: $MQ_3 \rightarrow P_3$.

The other converses, 4 and 5, can be derived in a similar manner.

The above proof can be applied to a still more generalized Law of Converses:

If
$ABC \cdots P_1 \rightarrow Q_1$
$ABC \cdots P_2 \rightarrow Q_2$
$ABC \cdots P_3 \rightarrow Q_3$

then
$ABC \cdots Q_1 \rightarrow P_1$
$ABC \cdots Q_2 \rightarrow P_2$
$ABC \cdots Q_3 \rightarrow P_3$

It is of course assumed that P_1, P_2, and P_3, and Q_1, Q_2, and Q_3 obey the same restrictions as heretofore.

Applications of the Generalized Law of Converses

1. Logical Derivation of s.s.s. = s.s.s. from s.a.s. = s.a.s.

The power of the above law may be exhibited by assuming that the following three theorems have been proved in the usual manner and by noticing the number of theorems that may be derived from them by the use of logic alone.

I.	II.	III.
In triangles ABC and $A'B'C'$		
$a = a'$	$a = a'$	$a = a'$
$B = B'$	$B > B'$	$B < B'$
$c = c'$	$c = c'$	$c = c'$
$\therefore b = b'$	$\therefore b > b'$	$\therefore b < b'$

I is the familiar theorem

If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of a second triangle, the third side of the first is equal to the third side of the second.

In the above set of theorems, the constituents of the second condition, $B = B'$, $B > B'$, and $B < B'$ exhaust all possibilities and are mutually exclusive. The same may be said of the conclusions, $b = b'$, $b > b'$, and $b < b'$.

The generalized law of converses may, therefore, be applied to the theorems, with the following resulting converses:

IV.	V.	VI.
In triangles ABC and $A'B'C'$		
$a = a'$	$a = a'$	$a = a'$
$b = b'$	$b > b'$	$b < b'$
$c = c'$	$c = c'$	$c = c'$
$B = B'$	$\therefore B > B'$	$\therefore B < B'$

By applying the theorem *s.a.s. = s.a.s.* to IV the theorem *s.s.s. = s.s.s.* can be shown to be true. Thus the theorem *s.s.s. = s.s.s.* is not only a converse of the theorem *s.a.s. = s.a.s.* but can be derived from it *by logic alone.*[8]

2. Logical Derivation of a.s.a. = a.s.a. from s.a.s. = s.a.s.

By a technique similar to the one used in the preceding section, the theorem *a.s.a. = a.s.a.* may be derived from the theorem *s.a.s. = s.a.s.*, by the combined application of the Law of Converses and the congruence theorem *s.s.s. = s.s.s.*

[8] See pp. 110–111.

3. *Application to a Theorem Having Four Data in the Hypothesis*

The theorem that "any point on the bisector of an angle is equidistant from the sides of the angle" can also be used as a starting point for a set of three theorems and their converses:[9]

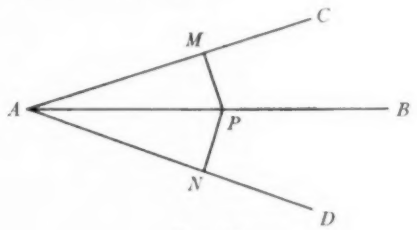

FIG. 12.

I.	II.	III.
$PM \perp AC$	$PM \perp AC$	$PM \perp AC$
$PN \perp AD$	$PN \perp AD$	$PN \perp AD$
AB bisects angle A	AB bisects angle A	AB bisects angle A
P is on AB	P is above AB	P is below AB
$\therefore PM = PN$	$\therefore PM < PN$	$\therefore PM > PN$

The converses of the foregoing theorems, obtained by interchanging the fourth condition in each hypothesis with the conclusion follow necessarily from I., II. and III. by the Law of Converses.[10]

4. *Application to a Theorem About a Perpendicular Bisector*

By an analysis similar to the above it can be shown that the theorem

Any point on the perpendicular bisector of a line is equally distant from the ends of the line

is the first of a set of six theorems, such that if the first three be proved true, their converses are also true by the Law of Converses.

NECESSARY AND SUFFICIENT CONDITIONS

Scarcity of Explanations of the Above Terms in Mathematical Textbooks

Calculus. It is certainly unnecessary to point out to the readers of this essay that a thorough grasp of the meaning of the concepts, *necessary and sufficient conditions*, is indispensable for an understanding of mathematical literature. Yet, of the textbooks in calculus examined, only two contained an explanation of these terms.[11] All the textbooks contained, however, theorems in which the concepts *necessary and sufficient conditions* were used, especially in connection with tests for the convergence of a series.

Analytic Geometry. Only one of the textbooks in analytic geometry examined contained the above terms.[11] In the theorems dealing with the conditions for perpendicularity and parallelism of lines, the authors evaded the use of the phrase "necessary and sufficient condition" by quoting the theorem and its converse. Thus

If two lines are parallel their slopes are equal, and conversely; if they are perpendicular their slopes are negative reciprocals, and conversely.

This formulation is, of course, correct, but it does not enable the student of analytic geometry to get acquainted with important logical concepts which he will inevitably encounter in his later courses in mathematics. Moreover, the full significance of the phrase "and conversely" usually escapes the student when he learns the above theorem. The emphasis on "and conversely" tends to hide the fact that the phrase "and inversely" might have been substituted for the commoner expression with greater benefit to his understanding. The phrase "and inversely" when expanded in the first part of the above theorem, means:

If two lines are not parallel, their slopes are not equal.

This expression, the writer believes, is

[9] The diagram in the text applies to I only. The words *above* and *below* may, of course, be changed to *to the right of* and *to the left of* depending on the position of the diagram.

[10] Another application, similar to the above, may be found in Upton, *op. cit.*, p. 119.

[11] Osgood, W. F., *Introduction to Calculus*, p. 71. The Macmillan Company. 1922. Love, C. E., *Differential and Integral Calculus*, p. 255. The Macmillan Company. 1934. Nowlan, F. S., *Analytic Geometry*, p. 223. McGraw-Hill Book Company. 1934.

more forceful and perhaps more meaningful than its logical equivalent, "and conversely."

Geometry. Of the geometry textbooks examined only a few contained the terms "necessary and sufficient conditions."[12] Unfortunately these important ideas were not treated as an essential part of the books, and could not therefore become a part of the students' vocabulary and way of thinking.

Difficulties in Understanding the Concepts of Necessary and Sufficient Conditions

Insufficient Opportunities for the Use of the Terms

As the foregoing survey indicates, the terms are not used with a frequency sufficient to make them an integral part of the students' mathematical vocabulary. If the terms are mentioned in geometry at all it is only in connection with problems on loci —surely an unfortunate coincidence. The following questions will indicate the possibilities of an early introduction of the concepts into geometry:

1. Is the statement that "two angles are vertical" sufficient for the inference of the statement that "two angles are equal"? Is it necessary?

2. Is the statement "If two adjacent angles have their exterior sides in a straight line" necessary for the statement "The two angles are supplementary"? Is it sufficient?

3. Is the statement "If a line is perpendicular to a radius at its outer extremity" necessary or sufficient, or both, for the statement, "The line is tangent to the circle"?

[12] Farnsworth, R. D., *Plane Geometry*, p. 136. McGraw-Hill Book Company. 1933. Halsted, *op. cit.*, p. 56. Hart, C. A. and Feldman, D. D. *Plane Geometry*, p. 47. American Book Company. 1911. McMahon, *op. cit.*, pp. 57–58. Solomon, C. and Wright, H. H., *Plane Geometry*, p. 155. Charles Scribner's Sons. 1929. Smith, D. E., Reeve, W. D., and Morss, E. L., *Exercises and Tests in Plane Geometry.* Tests 24, 42, 49, and 76. Ginn and Company. 1928. Swenson, J. A., *Integrated Mathematics with Special Application to Geometry*, pp. 27–28. Edward Brothers. 1934.

It should be noted that in the above examples the word *statement* was used in place of the more common word *condition.* The transition to the latter term will come easily enough if sufficient practice is afforded.

Elementary algebra, too, offers an opportunity for the use of those important concepts. Is the condition that "the discriminant of a quadratic equation be positive," necessary, sufficient, or both, for the condition that the "roots of the equation be real"?

Although it is not suggested that the concepts—necessary and sufficient conditions—be introduced into arithmetic, it affords opportunities for their use. The following question will illustrate the possibilities: What are the necessary and sufficient conditions for the divisibility of a number by 2? by 9?

Incomplete Realization of the Logical Law Involved

No serious student of the subject will deny that the lack of acquaintance with the logical concepts and their applications contributes greatly to the difficulty of understanding them. A more important factor, however, is the incomplete realization on the part of the textbook writers of the logical law underlying the notions of sufficient and necessary conditions. Thus many textbooks mention the adjective "necessary" first when enunciating a theorem that the condition A is a necessary and sufficient condition for B, but in the actual proof of the theorem they first prove that A is sufficient for B. This inconsistency between the order of the conditions given in the theorem and the order of proving them has undoubtedly been a source of confusion to many students of mathematics.

Another disturbing factor found in the treatment of this topic is the almost universal conspiracy of silence in textbooks on this important question: How does the truth of the statement, B implies A, justify the conclusion that A is necessary for B?

A third source of trouble is the tendency of nearly all books to state that in order to demonstrate that A is necessary for B, one may either prove that B implies A, or that not-A implies not-B without taking the trouble to prove, to explain, or at least to justify the implicit assumption that those two statements are equivalent to each other.

Preoccupation with the Words Necessary and Sufficient

In some discussions[13] of this topic too much emphasis is laid upon the words *necessary* and *sufficient*, instead of upon a definition of those terms which would lead to correct usage and interpretation. Although it is pedagogically desirable to derive from the general meaning of a term its special significance in a certain science, it is, however, a mistake not to realize that the technical meaning must ultimately be established by definition, and it is that definition that fixes its meaning in that science. Thus, the terms "complementary angles" and "vertical angles" in geometry, and "force" and "power" in physics derive their significance for the mathematician and physicist from the technical definitions given those terms, and not from the meaning they have in everyday life.

Fortunately the phrases "necessary condition" and "sufficient condition" are capable of exact logical definition, which would thus diminish the danger of any adventitious associations with the common meaning of the words "necessary" or "sufficient."

Application of the Law of Contraposition to the Phrase "Necessary and Sufficient Conditions"

Emphasis on Inverse Instead of Converse

It was pointed out[14] that it is of great importance even in elementary mathematics to stress the concept of the inverse of a proposition and its independence of the truth of the original statement. The Law of Contraposition emphasizes the equivalence of a converse and the corresponding inverse. With the help of these relations, the phrase—necessary and sufficient condition—may be defined in the way described below.

Definition of "A is a sufficient condition for B." The expression "A is sufficient condition for B" is equivalent in meaning to the statement "A implies B." Consequently in order to prove that "A is sufficient condition for B," one may follow one of the two following procedures:

1. Either prove that $A \rightarrow B$, or
2. Prove its contrapositive equivalent, $\overline{B} \rightarrow \overline{A}$.

Thus in order to establish that the equality of slopes is a sufficient condition for the parallelism of two lines, one of the following theorems will have to be proved:

1. If the slopes of two lines are equal, the lines are parallel.
2. If two lines are not parallel, their slopes are not equal.

Although in this case, as in many cases, the proof of the original theorem is easier than that of its contrapositive, the possibility of an alternative proof should always be kept in mind.

Definition of "A is a necessary condition for B." The expression "A is a necessary condition for B" is defined as synonymous with the logical expression $\overline{A} \rightarrow \overline{B}$.[15]

The above definition has the advantage of being in consonance with the common meaning of the word "necessary" as exhibited in the following expression: "Water is necessary for the growth of plants." Al-

[13] See Christofferson, *op. cit.*, pp. 125–128. Also Garabedian, C. A. "Some Simple Logical Notions Encountered in Elementary Mathematics." *The Mathematics Teacher*, 24: 345–352. October 1931.

[14] See page 162

[15] Ladd-Franklin, Christine, "Necessary and Sufficient Condition," *Dictionary of Philosophy and Psychology.* (J. M. Baldwin, Editor.) Vol. II, p. 143. Chapman, F. M. and Henle, P., The Fundamentals of Logic, pp. 312–313. Stebbing, L. S., *op. cit.*, p. 271. Cohen, M. R. and Nagel, E., *op. cit.*, p. 388. Eaton, R. M., *op. cit.*, pp. 521–522.

though it is positive in form, the intent of the above statement is unquestionably negative in character: "If no water is supplied, plants will not grow." The realization that a necessary condition really has negative implications is of utmost importance for clear thinking. To emphasize this negative character it has even been suggested that the phrase "indispensable condition" should replace the more common expression, "necessary condition."[16]

To prove that A is necessary for B, one may prove either

1. $\overline{A} \rightarrow \overline{B}$, or its contrapositive equivalent,
2. $B \rightarrow A$.

Thus to prove that the equality of the slopes of two lines is a necessary condition for their parallelism, either one of the following two theorems must be proved:

1. If the slopes of two lines are not equal, the lines are not parallel.
2. If lines are parallel their slopes are equal.

Advantages of the above approach. The procedure indicated above has the following advantages:

1. The vague meaning of the terms "necessary and sufficient condition" are replaced by exact, logical definitions.

2. The expression "P is necessary for Q" is defined in terms of the inverse proposition "$\overline{P}$ implies $\overline{Q}$," which is more in accord with the common sense idea than the converse proposition "Q implies P."

3. The baffling mystery of books sometimes proving the converse of a proposition and at other times its inverse in order to establish a necessary condition is resolved by the Law of Contraposition, which enunciates the equivalence of a converse and its corresponding inverse.

The method of contraposition as a substitute for the indirect method of proof. In addition to the propositions proved in the foregoing pages, the author found many more theorems and originals that are usually proved by the indirect method and that have yielded to simpler proofs by the method of contraposition. The temptation to generalize about the equivalence of the two methods of proof is very strong, but it will have to be resisted until a rigorous proof of such equivalence is found. On the basis of the empirical evidence available it is safe to form the tentative hypothesis that, in geometry at least, the method of contraposition seems to be an effective substitute for the indirect method of proof.

It is important to observe that although the method of contraposition can replace the traditional indirect methods of proof it does not eliminate the need for indirect reasoning; on the contrary, it is another method of indirect reasoning.

Application of the method of contraposition to other branches of mathematics. Preliminary investigations conducted by the writer in collaboration with Dr. Aaron Bakst of Teachers College, Columbia University, seem to indicate that the method of contraposition may also be applied to the proofs of a few theorems in Higher Algebra, in the Theory of Functions and in Point Set Theory. A detailed exposition of these proofs will, however, be out of place in this study, since it is primarily devoted to the field of elementary geometry.

Application of the method of contraposition to the reasoning of everyday life. Simple examples of reasoning by contraposition can be found in everyday life. The following quotations will illustrate this point:

(a) If a person noticing that a flag is hanging limp on a pole says, "There is no wind today" that person has reasoned as follows:
"If the wind were blowing, the flag would be waving. But the flag is not waving, so the wind is not blowing."

(b) One is looking for a book with a green cover, and finding a book the cover of which is not green knows at once that the book is not the book

[16] Ladd-Franklin, Christine, "Implication and Existence in Logic," *The Philosophical Review*, 21: 646. November 1912.

wanted. Definitely put, the argument would be:

> The book wanted has a green cover. This book has not a green cover, so it is not the right book.[17]

Upton[18] tells of an incident in which he had to use indirect reasoning in order to convince the engineer that the valve on the steam radiator was defective. The following analysis will indicate how the method of contraposition could have been used in that non-mathematical situation. Let g represent the proposition "The valve is good"; let c represent the proposition "The valve is closed"; and let h represent the proposition "The radiator is hot." The engineer will readily admit the truth of the following proposition:

$$gc \rightarrow \bar{h}$$

which means that if the valve is good, and it is closed, the radiator will not be hot.

By the Law of Contraposition the following proposition may be validly derived:

$$ch \rightarrow \bar{g}.$$

Translated into words this proposition means that if the valve is closed, and the radiator is hot, then the valve is not good.

A similar analysis can be made from the contrapositive point of view of the other interesting illustrations that Upton gives of the use of indirect proof in life. But the one given above is sufficient to show that the symbols of the schematic proposition

$$h_1 h_2 h_3 \cdots h_n \rightarrow c$$

and of the contrapositives derivable from it can be vested with significant, non-mathematical interpretations. It is this fact that lends importance to the introduction of the concept and the Law of Contraposition into secondary mathematics, rather than the possibility of using it as a device for circumventing traditionally difficult and clumsy proofs.

[17] Smith, E. R. *Plane Geometry Developed by the Syllabus Method*, pp. 15–18. American Book Company. 1909.

[18] Upton, *op. cit.*, pp. 102, 128–131.

An Error in Logic in Books on Geometry

The following quotation is, perhaps, as cogent an argument for the inclusion of the Law of Contraposition into the elementary course in geometry as has been presented in this study:

> If a direct proposition and its opposite are true, the converse is true; and if a direct proposition and its converse are true, the opposite is true.[19]

It is strange that so well known a writer of textbooks as Wentworth should have been unfamiliar with the elementary logical principle that the truth of the opposite *alone* is sufficient to warrant the truth of the converse, and vice versa. It is stranger still that the above statement which was undoubtedly studied by thousands of teachers and students of mathematics went unchallenged and uncorrected for at least fifteen years in the subsequent revision and editions.[20]

The same error is repeated almost verbatim in Durell's textbook,[21] and, as late as 1927, in Bernard's, *Plane Geometry*.[22]

An error involving the same logical principle may be found in Heath's famous work on *Euclid's Elements*,[23] both in the original edition of 1908 and in the revised edition of 1926.

As De Morgan points out (Companion to the Almanac, 1849, p. 7) I.6 is a purely *logical* deduction from I.5 and I.18 taken together, as is I.19 also. . . . Let X denote the class of triangles which have two sides other than the base equal, Y the class of triangles which have the base angles A equals. Then we may call non-X the class of triangles having the sides other than the

[19] Wentworth, G. A. *Plane Geometry*, p. 15. Ginn and Company. 1888.

[20] *Op. cit.*, p. 5, editions of 1899, 1901 and 1903.

[21] Durell, F. *Plane and Solid Geometry*. p. 25. Charles E. Merrill Company. 1904, 1908, 1912.

[22] Bernard, D. M. *Plane Geometry*, p. 160. Johnson Publishing Company. 1927. For a cryptic allusion to the error in "some textbooks," see Schultze, *op. cit.*, p. 145, note.

[23] Heath, *op. cit.*, vol. I, p. 256 (both editions).

base unequal, non-Y the class of triangle having the base angles unequal.

Thus we have

All X is Y, [I.5]
All non-X is non-Y. [I.18]

and it is a purely logical deduction that

All Y is X (I.6) [I.6]

The following are the theorems referred to by De Morgan and Heath:

I.5: In isosceles triangles the angles at the base are equal to one another.

I.6: If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.

I.18: In any triangle the greater side subtends the greater angle.

I.19: In any triangle the greater angle is subtended by the greater side.

Since the error was committed by so eminent an authority as Heath, and since, moreover, it went uncorrected in the revised edition and was quoted in that form without comment in some manuals for teachers, it may not be amiss to analyze it in some detail.

It is evident by the very definition of contraposition that

"All non-X is non-Y"

is the contrapositive of

"All Y is X"

and is therefore equivalent to it by the Law of Contraposition. The statement "All X is Y" is therefore totally unnecessary for the derivation of "all Y is X."

Since Heath's analysis seems to be based on remarks made by De Morgan it is necessary to examine the original and to locate the source of the error.

The following is an exact reproduction of the quotation referred to by Heath:

I.18, I.19. The latter of these, and I.6, is a purely logical consequence of I.18 and I.5, following independently of the meaning of the terms.

The following note is added by De Morgan in connection with the words *meaning:*

If A, B, C be three propositions of which one and only one must be true; and if P, Q, R be of the same kind; and if A always give P; B, Q; and C, R; then P always gives A; Z, B: and R, C. This is the connecting logical proposition of I.6 and I.19 with I.5 and I.18.

The "connecting logical proposition" is of course the "Law of Converses." It should be noted, however, that in the first quotation De Morgan says that I.19 and I.6 are purely logical consequences of I.18 and I.5, meaning thereby that the former pair is a logical consequence of the latter pair. But Heath committed the fallacy of division and inferred that I.19 and I.6 are *each* logical consequences, of I.18 and I.5.

The proof of the Law of Converses given in the beginning of this chapter bears out this distinction. The converse Q_3 implies P_3 is obtained by operating on the two theorems,

P_1 implies Q_1
P_2 implies Q_2

No use is made in the proof of the fact that

P_3 implies Q_3

In this connection it is interesting to note that although De Morgan harped on the fact that Euclid failed to recognize those theorems that are logically equivalent, he himself failed to note that the Law of Converses which he helped to make known in England, is only a special case of the more general Law of Contraposition.

SUMMARY

In this chapter the history of the Law of Converses is outlined, and documentary evidence is submitted that Hauber anticipated De Morgan in its discovery. A generalization of the Law of Converses is made, (for the first time, it is believed) and its applications to geometry are pointed out. The importance of the notions of "Necessary and Sufficient Conditions" is discussed, and the application of the Law of Contraposition to the clarification of these concepts is indicated.

Conclusion and Recommendations

The Purpose of Teaching Geometry

It is commonly accepted among teachers of mathematics and philosophers of education that the most important reason for teaching geometry in the high school is the imparting of a feeling for logical proof and for the nature of deductive reasoning.

The real purpose of the subject is suggested by the word "demonstrative" rather than by "geometry." The mere utilities of geometry have already been acquired before the pupil begins, if he ever does, the work in what is to him an entirely new field—that of logical proof. Nowhere in his previous training, nowhere else in his elementary education does he come in close contact with a logical proof. . . . We use geometry rather than a course in pure logic because of the pupil's familiarity with the figures used in the former subject.[1]

The logical concepts and laws presented in this study should certainly be made an essential part of the course of demonstrative geometry, if the ideals set forth above are to be realized.[2]

Pedagogical Recommendations

Although pedagogic hints have been given throughout this essay, it may not be out of place here to give some general recommendations for the introduction into the classroom of the logical ideas advocated in this study:

1. The three concepts should be introduced as early as possible in the course. Even the axioms yield interesting transformations.

2. Let the students discover for themselves by experiment the equivalence between a converse and an inverse. The Law of Contraposition should be treated as a postulate.

3. The students should be encouraged to bring in statements and their transformation from fields outside mathematics.

4. The Law of Contraposition should be used not only as a means of proving theorems but also as a means for discovering new theorems.[3]

5. The terms "necessary and sufficient conditions" should be used wherever possible. They have ramifications beyond the classroom and beyond mathematics.

6. At all times it should be borne in mind that the teacher of geometry has greater opportunity than the teacher of any other subject to leave upon the pupils' minds an indelible imprint of the nature of valid and invalid patterns of reasoning.

[1] Smith, David Eugene and Reeve, William David, *The Teaching of Junior High School Mathematics*, pp. 229–230. Ginn and Company, 1927. For a collection of the statements of the aims and values of geometry, see Shibli, J., *Recent Developments in the Teaching of Geometry*, pp. 203–210. State College, Pa.: J. Shibli. 1932.

[2] For the opposite point of view see (a) Smith, D. E., *The Teachings of Geometry*, pp. 104–105. (b) Schultze, *op. cit.*, p. 147.

[3] The arguments presented in this essay, urging the use of the Law of Contraposition in indirect proofs, should not be construed as implying that that is the only procedure that may replace the traditional methods of demonstration. There is, indeed, another law in logic, (see p. 171, Theorems I′ and II′) which may be employed successfully whenever indirect reasoning is to be used. A detailed study of that approach is, however, outside the scope of this study.

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THE ART OF TEACHING

Substitutions, and the Order of Operations

By JOSEPH A. NYBERG *Hyde Park High School, Chicago*

THE article by Wallace B. Graham in the March issue of *The Mathematics Teacher* suggests two interesting questions: What is the rule for order of operations? How should a pupil be taught substitution in a formula?

The rule for the order in which operations should be performed is usually stated:

First perform the multiplications and divisions, and then perform the additions and subtractions.

This statement is not clear. I believe Walter Hart was the first to emphasize that the rule should read:

First perform all the multiplications (naturally, in order from left to right); then perform all the divisions; then the additions and subtractions.

The need for this clarification will be obvious from an example. Consider any relation of the type $a=bcd$; let us use the common formula $i=prt$. Then $p=i\div rt$ or $p=i\div r\times t$. Since $p=i\div(r\times t)$ and not $(i\div r)\times t$, we see that the multiplication must be performed before the division even though the division sign preceeds the multiplication sign.

Note how easily the meaning is made clear by using parentheses. In other words, why should it be necessary to teach anything about the order of operations? If I have never been taught the order of operations and some one should ask me: What is the value of $24\div 3\times 2$? I would be justified in answering: I refuse to answer until you make your meaning clear by writing your question in an intelligent manner. Haven't you ever heard of parentheses? Or, I might answer: If you will show me the actual problem which gave rise to your question I can tell what you mean. Of course the pupil needs to be taught that the expression is ambiguous and needs clarifying.

The necessity for teaching something about the order of operations is due chiefly to a thoughtless way of handling substitutions in a formula. Consider the problem:

Find p if $i=prt$ and $i=30$, $t=3$, $r=.05$

If we substitute these values directly into the formula, as should be done, then $30=p\times .05\times 3$ and we need not know anything about the rule of order. If we, foolishly, first solve for p and then substitute, then

$$p=\frac{30}{.05\times 3}$$

and again the rule of order is unnecessary.

Consider next the problem:

Find the value of $3a^2-5a$ if $a=-2$

If we substitute directly, as some pupils are taught to do, then

$$3a^2-5a=3\times -2\times -2-5\times -2$$

and I *may* need to know something about the rule of order (and I may also need many other things, such as, courage to attack a complicated expression, and "stick-toativeness"). I say *may* need because what I do with this discouraging array of numbers depends on my understanding of the meaning of an expression like $3a^2-5a$. If I have been taught that $3a^2-5a$ means the sum of the two terms $3a^2$ and $-5a$, then I know that I must find the value of each term then add the two terms. Hence

the above expression means the number $3\times-2\times-2$ added to the number -5×-2. Similarly, a quantity like $3a^2-4bc+\frac{5}{bc}$ means $3a^2$ added to $-4bc$ added to $\frac{5}{bc}$. And again I have no need for learning anything about the correct order of the operations. In fact, if in this expression, $a=-2$, $b=3$, $c=4$, the teacher should never teach, encourage, or even allow a pupil to substitute directly into the expression. A more sensible way of arranging the work would be something like this:

$$a^2=4 \qquad bc=12 \qquad \frac{5}{bc}=\frac{5}{12}$$

$$3a^2=12 \qquad -4bc=-48$$

Then $3a^2-4bc+\frac{5}{bc}=12-48+\frac{5}{12}$ and again I need know nothing about any rule of order; I only need to know what I am talking about.

This leads to another pertinent question: Why should a pupil be asked to find the value of a quantity like $3a^2-7ab+8b^2$? This problem is surely not invented (and worked in a way that no teacher would) merely to have an excuse for teaching the rule of order. These evaluation problems exist, I believe, for such reasons as (a), (b), (c), (d) below:

(a) To furnish a review of arithmetic. As such they are very poor material since they seldom involve anything but integers and omit the fractions and decimals. Moreover, this review can be had in more intelligent ways.

(b) To remind the pupil that $a, b, c, \ldots$ are really numbers and not merely letters of the alphabet. This aim could also be accomplished in more intelligent ways.

(c) To serve as a preparation for the exercises in which the sum of several polynomials will be checked by substituting a number for each letter. But the proof, for example, that the relation

$$(5a^2-3a+7)+(2a^2+9a-3)=7a^2+6a+4$$

is true for all numbers would be more convincing if we did not limit ourselves to testing the relation for only $a=2$ but tested it also for three or four other numbers such as $a=-2$, $a=-3$, $a=\frac{3}{4}$, $a=\sqrt{6}, \ldots$ and few teachers would ask a class to do this.

(d) To serve as a preparation for the work with formulas. In this case, why not work with the formulas themselves? When the pupil works with formulas he will need to know the meaning of a polynomial, namely, a polynomial is the *sum* of several terms. To find the value of the polynomial he must therefore find the value of each term and *add* the terms. Why *add* them? Because a polynomial is the *sum* of several terms.

While discussing substitutions I should like to call attention to some other matters. One way in which teachers develop in the pupil a dislike or prejudice against algebra arises in giving such directions as:

Use the formula $c=np$ to find the cost of 5 lb. of sugar at 6¢ a pound.

The pupil thinks: "The cost is 30¢ and I don't need any formula to find it. Algebra is just a hard way of doing the problems of arithmetic."

The textbook may even advise the pupil to *substitute* in the formula $c=np$ in order to solve the above problem and, as an illustration, writes

$$c=np \qquad c=5\times6=30$$

The object of such work is evidently to teach how a formula is used, and we must of course begin with a simple problem. But sometimes the illustrations are so simple that they prejudice our case. It would be better to postpone the discussion until a more complicated formula can be used so that the result is less obvious or else to state the directions as follows:

What rule of arithmetic do you use to find the cost of 5 lb. of sugar at 6¢ a pound? Write this rule as a formula.

Or the directions could be stated thus:

To make your work clear both to your-

self and to any other pupil who might wish to examine your work, write (a) what formula you intend to use, and (b) write down what you intend to do with the numbers before you do it. Thus, if finding the cost of 5 lb. of sugar at 6¢ a pound, write

(a) $c=np$ (b) $c=5\times6$

I am indebted to Thorndike's writings for that very useful idea: write what you intend to do before you do it. In an article in the Seventh Yearbook of the National Council I showed how effectively this idea can be used in the teaching of verbal problems. When used with formulas it helps us to get what we want from the pupil without making algebra seem like arithmetic made difficult.

One more remark about substitutions. In the third and fourth year classes it is easy to detect the pupils who have been taught to substitute mechanically in formulas. When using the law of tangents in a trigonometry class they write

$$\tan \tfrac{1}{2}(A-B)= \frac{876.5-345.6}{876.5+345.6} \tan \tfrac{1}{2}(75\ 13'+29\ 32')$$

The pupil is of course blindly following the rule of writing what he intends to do instead of going ahead and doing it. Which shows that no rule should be followed at all times. Here it is best to tell the class that many practices which are useful when learning a subject may be disregarded after the subject has been mastered.

The Tenth Yearbook

of

The National Council of Teachers of Mathematics

on

The Teaching of Arithmetic

was recently voted one of the sixty most outstanding contributions of 1935. It discusses the most important issues with reference to the teaching of arithmetic at the present time, such as the problem of transfer, the mechanistic approach versus the understanding approach; the place of the activity program, informal versus computational arithmetic, current classroom practice, etc. The writers are all outstanding authorities in the field.

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Teachers College, Columbia University, New York, N.Y.

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EDITORIALS

Loyalty to the National Council

WE HAVE repeatedly had letters from teachers of mathematics from various parts of the country who write somewhat as follows: "Since the high school library has now ordered *The Mathematics Teacher*, I wish to discontinue my subscription." Now we realize that the main purpose of the magazine will be served if it is read and appreciated by teachers of mathematics whether they pay for membership in the National Council or not. However, just imagine what would happen to *The National Council of Teachers of Mathematics* if we all took that attitude. The official journal would have to close up its business and what little money the Council is able to save from subscriptions and the sale of yearbooks would no longer be available. This would mean that our annual meetings would cease and that we could no longer help to subsidize such important studies as the one that is now being made by *The National Arithmetic Committee* of which Professor R. L. Morton of Ohio University is Chairman. This Committee is entirely financed by the National Council. Moreover, the National Council has been largely responsible for starting the *Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics on "The Place of Mathematics in Secondary Education."* The report of this Commission will constitute the Fourteenth Yearbook of the National Council in 1939.

The National Council is the only organization in this country primarily devoted to the improvement of mathematical education. Teachers should know not only this but also that, in a sense, their very positions in the schools may depend upon how well the Council does its work. To do this work well, to put mathematics back into its rightful place in the schools, the National Council must have the support of all teachers, who make any claim to being loyal to the cause they represent. Too many teachers today are apathetic about their duty to the subject they represent, but we believe they will respond when they have been made to realize how much depends upon their support. Let's all get behind an organized movement to obtain a membership of 10,000 by 1940. Mrs. Florence Brooks Miller and her group of state representatives are doing a great service, but not all of the pioneer work must be left to them. If each present member will obtain one new subscriber, we will have attained our goal of 10,000.

W.D.R.

Progressives or Essentialists?

Most teachers would probably have difficulty in definitely identifying themselves with the camp of either the so-called "Progressives" or the "Essentialists." After all, does it make so much difference to lovers of mathematics what our exterior labels may be? What we need more than anything else right now is a greater number of high character men and women so trained as to be able to devote themselves to the teaching of mathematics throughout their active careers. Then if we can all combine to improve the organization and teaching of mathematics in the schools, what difference does it make to which camp we belong? We are all in the same boat and we sink or swim together. Besides, if the content material presented to the pupils is what it ought to be, it is conceivable that the method used does not necessarily have to be either one or the other.

W.D.R.

◆ IN OTHER PERIODICALS ◆

By NATHAN LAZAR
Alexander Hamilton High School, Brooklyn, N. Y.

1. Bradley, A. Day, "Pennsylvania German Arithmetical Books." *Scripta Mathematica.* 5: 45–51. January, 1938.

An interesting description of the contents of textbooks in arithmetic that were in use among the Pennsylvanians of German descent, during the latter half of the eighteenth and the first quarter of the nineteenth century. Many of the facts stated should be of interest not only to the mathematical historian but also to the social historian of the United States. "It is of some interest to note the space given in these books to tables of foreign coins current in the United States, and to rules for the exchange of state currencies. It is well known that foreign coins in great number and variety continued to circulate long after the adoption of the Federal currency and that the money issued by the states remained in use as a medium of exchange and keeping accounts for many years."

2. Corliss, John J., "Solution of the Quadratic Equation by Means of Complex Numbers." *School Science and Mathematics.* 38: 256–258. March, 1938.

Although the solution presented does not fall within the ability or knowledge of the students in the ninth and tenth grades, it should be of interest to those who are studying advanced algebra or who are members of mathematics clubs.

3. Ewing, Galen Wood, "An Optical Illustration of Conic Sections." *School Science and Mathematics.* 38: 276–277. March, 1938.

The author shows how a common, inexpensive flashlight can be used to give accurate as well as dramatic illustrations of conic sections. Specific directions and interesting photographs are included.

4. Keyser, Cassius Jackson, "Benedict Spinoza." *Scripta Mathematica.* 5: 33–36. January, 1938.

"The first of a series of sketches being prepared for a portfolio of the Portraits of Famous Philosophers Who Were Mathematicians to be published by *Scripta Mathematica.*"

After a brief but adequate sketch of Spinoza's life and works, the author himself asks the question "Was Spinoza a great mathematician?" His reply is as follows: "If the term is to designate only such as have made important contributions to so-called 'pure' mathematics, the answer is No. But if it be applied also to those who have masterfully applied the mathematical method to no matter what kind of concrete subject-matter, the answer is Yes. For, in the domain of Ethics, the mathematical method became in Spinoza's hands a powerful instrument for both research and exposition."

5. McGiffert, James, "The Economy of Symmetry." *National Mathematics Magazine.* 12: 210–212. February, 1938.

Many interesting examples of economy in structure are given, culled both from the animate and inanimate worlds. The mathematical principles are explained which underly the construction of the wax cells by bees and the manufacture of economical containers by human beings. "Many mathematicians are interested only in teaching pure mathematics, but if they are instructing embryonic engineers, they will accomplish a great deal for their students if they would impress upon them the fact that every object has its own most perfect dimensions for economy, as well as for efficiency."

6. Miller, Norman, "The Role of Definitions in High School Mathematics." *The School, Secondary Edition* (Toronto, Canada) 26: 227–235. November, 1937.

The author points out the importance of having exact definitions of the concepts used in secondary mathematics, but also warns of the dangers in the way of attaining that desirable objective. Specific illustrations are taken from algebra, geometry and trigonometry. The important distinction is made between the elucidation of the *meaning* of a technical term and the *formal definition* of that term.

7. Purdy, C. R., "Round Table Discussions in Arithmetic Work." *School Science and Mathematics.* 38: 258. March, 1938.

The author describes the technique of "Round Table Discussion" that he has found very effective in teaching certain topics to classes in arithmetic.

"Nearly every unit of Junior High School work has topics that lend themselves to the method. Here are a few of the topics that we have tried: sales briefs used by unscrupulous merchants, calendar reforms, ways of cooperation in the family budget, cost of medicine and socialized medicine, cost of an education, best methods for investing savings, and numerous others."

8. Read, Cecil B., "Further Study of the Relative Efficiency of Two Methods of Approximating the Roots of an Algebraic Equation." *School Science and Mathematics*. 38: 264–270. March, 1938.

The two methods of approximating the roots of an algebraic equation that were studied were the following:

a. Horner's Method
b. The Method of Successive Graphs

The author states that "no claim is made that absolutely conclusive evidence is yet available. Corroborating the results of the previous study, it may be stated with even greater certainty that if two or three decimal place accuracy is sufficient, a solution by successive graphs enables either the beginning or the advanced student to obtain the desired answer with less expenditure of time than Horner's method. Should accuracy to more than three decimal places be required, it is of importance to note that the graphic method is no longer the more efficient. . . . "

The article under review is based on an unpublished doctor's field study. The "previous study" mentioned above appeared in *School Science and Mathematics*, January, 1935.

9. Swenson, John A., "Algebraic Treatment of Euclid's Geometry." *School Science and Mathematics*. 38: 323–327. March, 1938.

As is well known, Dr. Swenson believes that students of mathematics should be taught to use the powerful tool of analytic geometry as early as possible. In this article he outlines in some detail the procedure to be followed in introducing the beginning student to the concepts and techniques of analytic geometry. He also shows how the important theorems of geometry, dealing with congruence, parallelism, and similarity can be approached from that point of view. The theorem of Pythagoras is proved by means of trigonometric functions. "This last theorem yields the distance formula in two dimensions, and this formula is used to solve loci problems. The algebra is introduced as needed and carefully constructed on the fundamental laws. . . . While practically no attention is given to puzzle problems, the problems which deal with the decimal system and commercial applications are amply dealt with. Extensive use of decimals, fractions, and percents is made throughout."

10. Yates, Robert C., "An Ellipsograph." *National Mathematics Magazine*. 12: 213–215. February, 1938.

A note and a diagram explain the construction of a linkage which will mechanically describe an ellipse and at the same time produce straight line motion.

Notice to Subscribers

NOTICES have recently been sent out to all subscribers whose subscriptions expire in May asking them to send in their renewals at once so as to save both them and *The Mathematics Teacher* unnecessary trouble and expense. Already letters are coming in to us from some of our subscribers saying that since their subscriptions began in October, they do not see why they should be notified now. For the sake of clearness, we must emphasize the fact that the magazine does not appear in June, July, August or September and hence a subscription which begins with the October issue expires with the eighth issue—May—and similarly for a subscription beginning with any other month. Only eight issues are given for $2.00.

Most of our subscribers are sending in their renewals promptly and this is a great help to us. We would urge all subscribers, who have not done so, to read the material on the inside front cover of this issue. Some important and useful material is given there.—The Editor.

NEWS NOTES

Of the three national organizations devoted to the promotion of mathematics, the oldest is the American Mathematical Society, which this September is celebrating the fiftieth anniversary of its founding in 1888. The Mathematical Association of America, which usually holds its summer meeting in conjunction with that of the Society, this year is omitting its meeting altogether and inviting its membership to share in the celebration of the older organization. It is the desire of the Society to extend an invitation to participate also to the membership of the National Council of Teachers of Mathematics.

Within the indivisible realm of mathematics, convenience has dictated the formation of three separate organizations—the National Council of Teachers of Mathematics, to deal with the field of elementary and secondary mathematics; the Mathematical Association of America, to cover the collegiate field; and the American Mathematical Society, to promote mathematical research and the interests of the universities. But although the welfare of one is the welfare of all it sometimes seems that this community of interest is not so generally recognized as it might be. The research mathematician, preoccupied with his work and happy in the satisfaction that American mathematics is recognized as being in a position of world leadership, does not know, or at least, fully realize the seriousness and extent of the attacks upon mathematics which his colleagues of the schools are now sustaining, and the school mathematicians, in the pressure of their everyday tasks, have failed to sense the tremendous advance which their university colleagues have effected along the mathematical frontier, and the possible significance of these advances in developing public appreciation of the values of mathematics. It is earnestly to be hoped that this American mathematical jubilee may contribute to the breaking down of artificial barriers between the groups.

In its younger days, of course, the American Mathematical Society was less restricted in its interests. The definitions of the requirements in secondary mathematics which dominated instruction from 1903 until the publication of the Report of the Committee on Reorganization of Secondary Mathematics in 1923 was prepared by a committee appointed in 1902 by the American Mathematical Society. In passing it may be noted that the 1923 committee was appointed by the Mathematical Association of America, an organization formed in 1916 with the full consent and encouragement of the Society—which wished to be free to devote itself to its primary aim of fostering mathematical research, and that the committee presently working—the Joint Commission on the Place of Mathematics in the Secondary Schools—was appointed cooperatively by the Association and the National Council.

It is hard to believe that the oldest national American mathematical organization has had but a half century of existence, and that its organization fifty years ago was brought about by a young college instructor rather than by the older and better-known mathematicians of the day. Dr. Thomas Scott Fiske, of Columbia College, recently returned from study in England, was moved because of the inspiration which had come to him at the meetings of the London Mathematical Society, to establish a similar society for the discussion of mathematical topics in New York. From the half dozen who gathered on a November day in 1888 to form the New York Mathematical Society, the strong and respected American Mathematical Society of today with its nearly two thousand members has grown.

Publication of contributions to knowledge in its field is one of the chief functions of a learned society. Since 1891, when the "Bulletin of the New York Mathematical Society" was established until today the Society has carried on this task. The "Bulletin of the American Mathematical Society," as it has been known since 1894, reports official acts of the Society, and publishes addresses delivered at its meetings, notes of current interest, short research papers, and reviews of advanced mathematical books; the "Transactions of the American Mathematical Society," devoted exclusively to research papers, was established in 1900; and the American mathematical research journals published under other than Society auspices all include official representatives of the Society on their editorial boards. The twenty-four volumes of the Colloquium series, each of which is an authoritative monograph on the recent developments of a restricted field of mathematics, constitute a contribution of no little significance to the mathematical world.

The Society holds ten or more meetings annually at which approximately five hundred research papers are presented, conducts courses of lectures on recent mathematical developments, maintains the Josiah Willard Gibbs pub-

lic lectureship on applied mathematics, owns an exceptionally strong library, open to its members, and administers several endowed prizes for excellence in mathematical research. The Society is acting as American host to the forthcoming International Congress of Mathematicians at Cambridge, Massachusetts, in 1940.

These activities of the Society but reflect its constant emphasis upon the importance of mathematical research, its unceasing encouragement to those participating in this work, and its unremitting maintenance of the highest standards of accomplishment. The Society is undoubtedly the greatest single force which has brought American mathematics to its present high position.

It is a far cry from these glories to the difficulties of early pioneer days; it is a source of gratitude on the part of the Society that Doctor Fiske, the principal founder, is serving today as chairman of the Committee arranging the Semi-Centennial celebration, to which the September meeting of 1938 is largely given over. These meetings will be held at Columbia University, New York City, from Tuesday to Friday, September 6–9, 1938.

The celebration proper will begin on Tuesday evening, when Doctor Nicholas Murray Butler, President of Columbia University, will deliver an address and representatives of learned societies more or less closely associated with mathematics will present greetings. The next morning Professor Raymond Clare Archibald of Brown University will survey the history of the Society, and Dean George D. Birkhoff of Harvard University will analyze American contributions to mathematics during the past fifty years. Probably one of the most delightful features of the celebration will be the Birthday Dinner on Wednesday evening at the Hotel Astor, an affair which is to be gay and gala rather than serious and significant.

The principal scientific feature of the jubilee will be the presentation of nine distinguished addresses by American mathematicians, each surveying a particular field of mathematics, with especial reference to the past fifty years and the work of Americans. The speakers, many of whom are still young men, are: E. T. Bell, California Institute of Technology; G. C. Evans, University of California at Berkeley; E. J. McShane, University of Virginia; J. F. Ritt, Columbia University; J. L. Synge, University of Toronto; T. Y. Thomas, Princeton University; Norbert Wiener, Massachusetts Institute of Technology; R. L. Wilder, University of Michigan.

Social events, in addition to the dinner, include a sail up the Hudson River to West Point on a steamer chartered for the occasion, with a battalion review at the United States Military Academy; a special program at the Hayden Planetarium, and provision for trips to the many museums and points of interest in New York City.

A very cordial invitation is extended by the American Mathematical Society to the members of the National Council of Teachers of Mathematics and to their friends to participate in this celebration on precisely the same terms as members of the Society itself. Those attending the meeting and their families may stay in the Columbia dormitories for all or part of the week. Copies of both the preliminary and final programs, as well as all necessary information, will be gladly sent upon request addressed to Miss Evelyn M. Hull, Office Manager, American Mathematical Society, 531 West 116 Street, New York City.

The Mathematics Club of Greater Cincinnati was organized at meetings held January 22 and February 19, and within a few days one of its charter members, Dr. C. H. Christofferson of Miami University, Oxford, Ohio, was elected president of the national body. That impetus should carry the club a long way. Dr. Christofferson will give a report on the national convention at the meeting to be held at noon May 21.

Teachers from southwestern Ohio were invited to the organization meetings, and there was the gratifying response of fifty-four charter members. At the January meeting the officers elected were: President, Mr. Cecil Tower; Vice President, Miss May Conner; Secretary, Miss Alma Wuest; Treasurer, Mr. Denzil Carpenter.

The February meeting was addressed by two men well qualified to speak on mathematics in industry. Mr. Harry C. Fischer, chemist of the Richardson Paper Company, showed that mathematics is used in every process of paper manufacture. Mr. Everett B. Woodruff, the Results Engineer of the Columbia Power Station of the Cincinnati Gas and Electric Company, said that mathematics is the keynote of the power industry and that everyone connected with a power station must use mathematics. He gave instances of some calculations that stationary engineers and foremen must make.

At the same meeting Mr. Walter S. Carnahan, President of the flourishing Mathematics Club of Indianapolis made some interesting and pertinent suggestions for keeping up interest and attendance.

On October 28 the Club is joining with the Mathematics Section of the Southwestern Ohio Teachers' Association in a luncheon meeting at Walnut Hills High School at which Dr. Reeve

will be the speaker. There will also be a display of textbooks. In the evening the Club will have a dinner at which Dr. Reeve will conduct a discussion. This will be such an important occasion that members of other mathematics clubs are invited to be present. Reservations are to be sent to the secretary, Miss Alma Wuest, 158 West Mc Millan St., Cincinnati, Ohio.

MRS. SUSAN MC LAUGHLIN
Hughes High School,
Cincinnati, Ohio.

Report of the Work of the State Representatives of the National Council of Teachers of Mathematics, January 1938

This report has to do with:

I. The personnel of the State Representatives and their work.

II. The membership of the Council in each state shown by
  (a) Number of members in each state
  (b) Number per thousand teachers in each state
  (c) List of states in order of membership
  (d) List of states in order of number per thousand teachers in each state

III. Financial statement.

I. Since the last annual meeting there has been a change in Representatives in the following states: Arkansas, Georgia, Idaho, Nevada, Ohio, Wisconsin, and Wyoming. In Washington, D.C. there have been two Representatives, one for the colored schools and one for the white. The latter, Mr. Wallis, is retiring from school work, so someone will have to be appointed in his place. When that is accomplished we shall have a complete list again.

With very few exceptions, the Representatives have responded to letters, to requests for information and have reported on plans used in promoting the welfare of the Council in the state. A file, containing all of the correspondence between the Representatives and the Chairman, has made it possible to compile information such as, the time of education meetings in the various states, and the types of publicity resorted to by different Representatives. On an extra sheet will be found an up-to-date list of the State Representatives and their addresses.

II. Comparison of membership by states would be more fair if represented as per thousand teachers of mathematics in the state. An attempt was made to secure from the Representatives the number of mathematics teachers in each state. A state directory of teachers arranged by subjects taught, is too much to hope for, in each state. Eight states were heard from but the information was too inexact to be useful.

Through the Director of Research of the N.E.A. Mr. William G. Carr, information was received concerning two studies which were thought might be helpful. One of these was, "A Summary of the Number and Percent of Teachers in Each Field of the Secondary Schools—A Consolidation of 25 Studies. September, 1936." The other was, "Distribution of High School Teachers by Major Teaching Fields, May, 1937."

The first of these studies involved 98,464 High School Teachers and 17% of them were teachers of mathematics. The second included 227,727 teachers not counting 7th and 8th grade teachers in the Junior High School, and there were 13% of them teaching mathematics. This information is too indefinite to be used in finding the number of mathematics teachers in each state.

On page 190 of the September, 1937 N.E.A. Journal there is a list of the number of teachers in each state. For making a comparison of the membership in the different states, it has been decided to use these numbers as a basis in determining the ratios of Council membership in the states. The accompanying tables are self-explanatory.

III. *Financial statement.* At the annual meeting last year the Board gave $100 to be used to help meet the expense connected with the work of the State Representatives. An appeal to the Budget Committee brought about an increased allotment, making $250 in all.

In distributing money to the Representatives, it was kept in mind that those in the vicinity of the meeting places of the Council should have more money to be used in publicity work.

The Representatives of the following states received $3: Alabama, Arkansas, Arizona, California, Florida, Connecticut, Georgia, Idaho, Kentucky, Louisiana, Maine, Massachusetts, Mississippi, Missouri, Montana, Nebraska, Nevada, New Hampshire, New Mexico, North Dakota, Oregon, Rhode Island, South Dakota, Utah, Vermont, Washington.

Those receiving $4: District of Columbia, North Carolina, Tennessee, Virginia, West Virginia.

Those receiving $5: Colorado, Delaware, Illinois, Indiana, Iowa, Kansas, Maryland, Minnesota, New York, Ohio, Oklahoma, Pennsylvania, South Carolina, Texas, Wisconsin, Wyoming.

Those receiving $10: Michigan, New Jersey.

May 1, 1937, Received	$100.00	
October 9, 1937, Received	150.00	
Sent to State Representatives		$201.00
Charges for checks by bank		2.20
Spent for stationery		7.00
Spent for file folders		.30
Postage		6.00
Received by money order from Wisconsin	1.80	
	$251.80	$216.50
Balance on hand, January 8, 1938		35.30
	$251.80	$251.80

Recommendations:

1. That the State Representatives be provided with more money to carry on the work which, in the majority of states, has been well started.

2. That stationery with the National Council heading be provided for the State Representatives. Such stationery would tend to make their correspondence more effective. A number of Representatives have asked for it. It might be purchased by the different Representatives, or it might cost less if ordered in large quantities at one place and distributed to them.

Respectfully submitted,

(Signed) FLORENCE BROOKS MILLER
January 7, 1938

State Representatives of the National Council of Teachers of Mathematics
January, 1938

Alabama—Mr. J. E. Allen, Phillips High School, Birmingham
Arkansas—Dr. Davis P. Richardson, Univ. of Arkansas, Fayetteville
Arizona—Miss Myra R. Douns, 93 West Culver, Phoenix
California—Miss Emma Hesse, University High School, Oakland
Colorado—Mr. H. W. Charlesworth, East High School, Denver
Connecticut—Miss Dorothy S. Wheeler, Bulkley High School, Hartford
Delaware—Mr. Edwin H. Downes, Dover
Dist. of Col.—Mrs. Ethel Harris Grubbs, 751 Fairmont Street, Washington
Florida—Dr. F. W. Kokomoor, University of Florida, Gainesville
Georgia—Dr. Eucebia Shuler, 312 West Church Street, Americus
Idaho—Mr. Norman Roberts, 101 South 1st, East, Preston
Illinois—Mr. C. M. Austin, Oak Park
Indiana—Dr. L. H. Whitcraft, Ball State Teachers College, Muncie
Iowa—Miss Ruth Lane, University High School, Iowa City
Kansas—Dr. U. G. Mitchell, University of Kansas, Lawrence
Kentucky—Miss Dawn Gilbert, 1331 Clay Street, Bowling Green
Louisiana—Mrs. H. L. Garrett, Box 427, University Station, Baton Rouge
Maine—Miss Pauline Herring, 360 Spring Street, Portland
Maryland—Miss Agnes Herbert, Clifton Park Jr. High School, Baltimore
Massachusetts—Mr. Harold B. Garland, 129 Houston Avenue, Milton
Michigan—Mr. Duncan A. S. Pirie, 950 Selden Avenue, Detroit
Minnesota—Mr. H. G. Tiedeman, Mountain Iron
Mississippi—Mr. Dowey S. Dearman, State Teachers College, Hattiesburg
Missouri—Mr. G. H. Jamison, State Teachers College, Kirksville
Montana—Miss Gertrude Clark, 403 Eddy Avenue, Missoula
Nebraska—Dr. A. R. Congdon, University of Nebraska, Lincoln
Nevada—Miss Bertha Knemeyer, Elko
New Hampshire—Mr. H. Gray Funkhouser, Phillips Exeter Academy, Exeter
New Jersey—Mr. Chalres D. Wildrick, Englewood
New Mexico—Miss Olive Whitehill, 402 West 8th Street, Deming
New York—Mr. H. C. Taylor, Benjamin Franklin High School, Rochester
North Carolina—Professor H. F. Munch, Chapel Hill
North Dakota—Miss Henrietta L. Brudos, State Teachers College, Valley City
Ohio—Mr. Hale Pickett, Athens
Oklahoma—Miss Eunice Lewis, 208 South Birch, Sapulpa
Oregon—Mr. Edgar E. De Con, University of Oregon, Eugene
Pennsylvania—Dr. C. N. Stokes, Temple University, Philadelphia
Rhode Island—Mr. W. Wilson Talley, Mary C. Wheeler School, Providence
South Carolina—Miss Hortense Rogers, Winthrop College, Rock Hill
South Dakota—Mr. Leroy Rowso, Miller
Tennessee—Dr. F. L. Wren, Peabody College, Nashville
Texas—Miss Elizabeth Dice, North Dallas High School, Dallas
Utah—Mr. Edwin M. Bronson, West High School, Salt Lake City
Vermont—Mr. G. H. Nicholson, 328 North Avenue, Burlington
Virginia—Mr. L. G. Lankford, Jr., University of Virginia, University
Washington—Miss Kate Bell, The Lewis and Clark High School, Spokane
West Virginia—Miss May L. Wilt, 107 High Street, Morgantown
Wisconsin—Miss Marie Chrisler, Kenosha High School, Kenosha
Wyoming—Dr. O. H. Rechard, University of Wyoming, Laramie

Membership in the National Council of Teachers of Mathematics as of November, 1937, by states. Also the number of members per thousand teachers of all subjects, by states.

States	*No. Teachers in State*	*Members of the N.C.T.M.*	*N.C.T.M. Members per 1,000 Teachers*
Alabama	18,341	53	2.88
Arizona	3,497	15	4.28
Arkansas	12,699	34	2.67
California	40,428	187	4.62
Colorado	9,277	59	6.35
Connecticut	10,589	80	7.55
Delaware	1,716	12	6.99
Washington, D.C.	2,935	63	21.46
Florida	12,409	80	6.45
Georgia	20,783	46	2.21
Idaho	4,588	8	1.74
Illinois	47,179	573	12.14
Indiana	21,983	161	7.32
Iowa	25,107	152	6.05
Kansas	19,455	174	8.94
Kentucky	17,833	59	3.31
Louisiana	13,210	34	2.57
Maine	6,119	33	5.39
Maryland	8,536	72	8.43
Massachusetts	27,583	273	9.89
Michigan	33,027	346	10.47
Minnesota	21,913	119	5.43
Missouri	24,199	126	5.20
Mississippi	14,912	62	4.16
Montana	5,648	23	4.07
Nebraska	13,989	94	6.72
Nevada	870	2	2.30
New Hampshire	2,949	24	8.14
New Jersey	27,094	231	8.53
New Mexico	4,500	13	2.89
New York	78,512	700	8.92
North Carolina	23,455	99	4.22
North Dakota	8,121	18	2.24
Ohio	42,896	302	7.04
Oklahoma	18,856	133	7.05
Oregon	7,568	44	5.81
Pennsylvania	62,221	437	7.02
Rhode Island	4,479	32	7.31
South Carolina	13,072	58	4.43
South Dakota	9,024	16	1.77
Tennessee	19,348	103	5.32
Texas	44,523	179	4.02
Utah	4,886	13	2.66
Vermont	2,944	36	12.23
Virginia	16,974	64	3.77
Washington	11,476	64	5.58
West Virginia	16,200	61	3.77
Wisconsin	21,852	152	6.96
Wyoming	2,678	14	5.23

Two other tables including the arrangement of states in order of their membership per 1,000 teachers in the states and the states in order of their membership in the National Council were omitted in duplicating this report because of lack of space and because of their being a repetition of the above except a difference in arrangement of the names of the states.

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NEW BOOKS

An Introduction to the Theory of Statistics. By G. Udny Yule and M. G. Kendall. J. B. Lippincott Company, 1937, XIII+570 pp. Price, $12 less 20% to teachers, educational institutions, and libraries.

This new book is a revised edition of an earlier edition (1912) by Mr. Yule. The complete revision has enabled the authors to produce a book of larger page and type than that of the older edition. This is a distinct improvement.

Apart from some substitution of new numerical illustrations to replace old, very little of the material of the older editions has been omitted. The major changes consist in additions of which the most important are several new chapters on Sampling, including an introductory chapter on Small Samples. Chapters have also been added on Moments and Measures of Skewness and Kurtosis, and on Simple Curve Fitting by the Method of Least Squares. Mr. Yule has contributed a new chapter on Interpolation and Graduation. For the first time Tables of the Various Functions commonly required in statistical work have been placed at the end of the book.

As in the earlier edition the authors have attempted "to keep within the limits laid down in regard to the knowledge of mathematics required by its readers" although they found it necessary in one or two places to introduce the notation of the integral calculus.

Teachers and students who used the earlier editions as well as others who are looking for good source material will be glad to see this book.

W.D.R.

Second Year Algebra. By W. A. Hawkes, H. E. Luby, and F. C. Touton. Ginn and Company 1938, VIII+498 pp. Price $1.48.

This new text is a revision of the enlarged edition of the authors' new Second Course in Algebra and is intended to follow their First Year Algebra and includes besides the material in their shorter Second Year Algebra, together with enough Advanced topics to furnish enough work and optional material for a complete second year of algebra.

In order to meet the increasing demand from teachers who may desire to give a simple introduction to some of the more advanced parts of mathematics, the authors have included two chapters which give an introduction to analytic geometry and the calculus. Obviously this material intended to attract those teachers whose students expect to take the so-called Gamma examinations of the College Entrance Examination Board.

As is the case with most books of this type a great deal of space at the beginning is given to a review of elementary algebra. Teachers who have been using the older editions of the same authors will find new book of interest.

W.D.R.

Socialized General Mathematics. By W. W. Hart and Cottell Gregory. D. C. Heath and Company, VII+406 pp. Price, $1.28.

This new book is intended to supply a course in quantitative thinking about socially useful material for those pupils who are not electing the traditional courses in mathematics in the first years of the high schools. It is general mathematics in the sense that it contains work in arithmetic, informal geometry, elementary algebra, and numerical trigonometry. It does not, however, attempt to introduce the pupil to demonstration.

The book is well written, attractively organized and has the advantage over other books intended to help the same class of pupils in that the formula method of solving problems is used to help the pupils to understand some of the more difficult parts of arithmetic.

If this book could be followed by a second course where an introduction to demonstration and some more of the important and useful phases of algebra might be included, it would form a fairly satisfactory training for most pupils who cannot or will not continue their study of mathematics. Perhaps this book will make those who study it eager to continue.

W.D.R.

Graphical Calculus. By Theodore R. Running. George Wahr, Ann Arbor, 81 pages, Price, $1.90.

This book is intended for those students who have had an elementary course in calculus and wish to use the graphical method in their treatment of data, particularly of experimental data.

Each one of the chapters is based upon the

principle of areas, and to the student who understands this principle the application of calculus in a number of ways is relatively simple.

The standing of the author assures that the content material will be useful.

W.D.R.

Social-Business Arithmetic. By W. S. Barnhart, and L. B. Maxwell. Mentzer, Bush and Company. VII+637 pp.

In this new book the authors claim to make a novel departure in the following respects:

1. Combining business arithmetic with the fundamental principles of business.
2. Organizing the book into chapter-units and lesson-units.
3. Planning the book so as to make necessary a good lesson assignment.
4. Giving especial attention to certain specific subjects such as use of the listing machine, public utilities in the home, occupational information, etc.
5. Providing differentiated lesson materials.
6. Making adequate provision for testing and remedial work.
7. Giving a thorough treatment of farm problems.

The book is well made, is attractively bound and will be of interest to teachers who wish to give a course of this kind.

W.D.R.

Experiences in Building a Curriculum. By Harold Spears. The Macmillan Co., 1937. XI+196 pp. Price, $2.

This book gives a wholly practical approach to the problem of curriculum construction. The program outlined is not final but rather elastic. Subject by subject in the Evansville (Indiana) high school used in this study were revised, correlated, and organized not with any idea of giving a finished product but a method of procedure. The book shows how the school and community cooperated in trying to solve some of the perplexing problems of the day.

Since curriculum making is one of the major interests of educators today this book will be of interest to all curriculum workers. The book is illustrated freely.

W.D.R.

Consumers Mathematics. By Anne Louise Cowan Stackpole Sons, 1938. XIV+324 pp. Price, $1.25 f.o.b. Harrisburg, Pa.

This book is intended to furnish "a Guidance Course in Commercial Relations." The fact is, such a course is needed or at least the kind of instruction to be given in such a course is much needed if people are not to continue to spend their money in foolish ways. In this book saving and buying are treated in a plain, practical and instructive manner. Taxes, communication, transportation and travel are presented in turn. Guidance in good citizenship and in individual thrift are predominant ideals of the book. One cannot help feeling as he goes through the various parts of the book that better understanding on the part of the pupils might be obtained by an algebraic treatment of some of the topics.

W.D.R.

The Classroom Teacher's Estimation of Intelligence and Industry of High School Students. By Harry Eisner, Bureau of Publications, Teachers College, Columbia University, 1937. Contributions to Education No. 726, 108 pp. Cloth. $1.60.

It is generally conceded that the most important determining factors in the successful achievement of high school students are their intelligence and their industry. This study, which is experimental in nature, was undertaken primarily to answer these questions: How well can the high school teacher estimate the intelligence and industry of his pupils on the basis of early classroom manifestations and responses? Are better estimates obtained when a special observational technique is employed? How closely do these estimates of intelligence and industry correlate with scholastic achievement and with final marks? The findings are of special interest to every high school teacher and supervisor.

W.D.R

Introduction to the Mathematics of Business. By W. L. Hart. D. C. Heath and Co., 1936. VII+317 pp. Price $2.60.

This text is intended to supply a course in applied mathematics for those students who may have only ordinary mathematical ability and training, but who have more than a passing interest in the quantitative aspects of the situation in which they find themselves and in the adult lives into which they are now entering. The main applications relate to simple interest, compound interest, annuities certain, pensions or life annuities, life insurance, and selected topics from the field of statistics. The book is the outgrowth of a three-year experiment at the General College of the University of Minnesota. The book is planned as a one year course meeting three hours per week. Any student who has had a year of elementary algebra and the general maturity of a college freshman is not expected to find the book too difficult.

The standing of the author in this field argues strength for any course which he has prepared.

W.D.R.

Elements of Statistical Method. By Albert E. Waugh. McGraw-Hill Book Co., 1938. XV+381 pp. Price, $3.50.

This new book is intended to give a simple, readable exposition of the basic principles and methods of statistics which are common to scientific research in all fields, without specialization in any branch to the exclusion of others. Emphasis is placed upon the understanding of the concept rather than upon mathematical manipulation.

It is intended for the beginner in the field of statistics who has to start "from scratch," and does not claim to exhaust any aspect of the field. For this reason students who wish to be introduced properly to statistical concepts and nomenclature and to begin to think in statistical terms should find this book helpful.

The type and general appearance of the book are good.

W.D.R.

Mathematics for Elementary Schools—A Handbook for Teachers. The University of the State of New York, 1937. 192 pages.

This monograph is the final report of a committee appointed by the State Department to revise the arithmetic syllabus of New York State. The major responsibility for preparing the final draft of the syllabus for publication fell chiefly upon Harris Crandall, F. Eugene Seymour, Vera Sanford, and Ben Sueltz (chairman). The revised syllabus shifts the center of emphasis from computation to the utilization of pupils' experiences as the chief material of instruction, from manipulation to developing an understanding of concept, from drill to mathematical thinking. Moreover, the syllabus is not strictly an arithmetic syllabus. It is an elementary mathematics syllabus giving a place to arithmetic, informal geometry, algebra and a simple approach to indirect measurement. The principle of social utility was employed in selecting the content material for the syllabus. As a result subject-matter that has outlived its usefulness has been replaced by material more in line with such economic movements as insurance, instalment buying, social security legislation and the like.

The syllabus is so planned as to help teachers to adapt their instruction to the needs of individual pupils. Moreover, it is meant to be flexible. However, it is clear that if the work is to be successful in the classroom, many arithmetic teachers will need to improve their backgrounds in elementary mathematics.

The State Department in New York is to be commended in securing such an excellent group of people to work on this syllabus. It is sure to bring about a better situation in mathematics in the elementary schools of the State of New York.

W.D.R.

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